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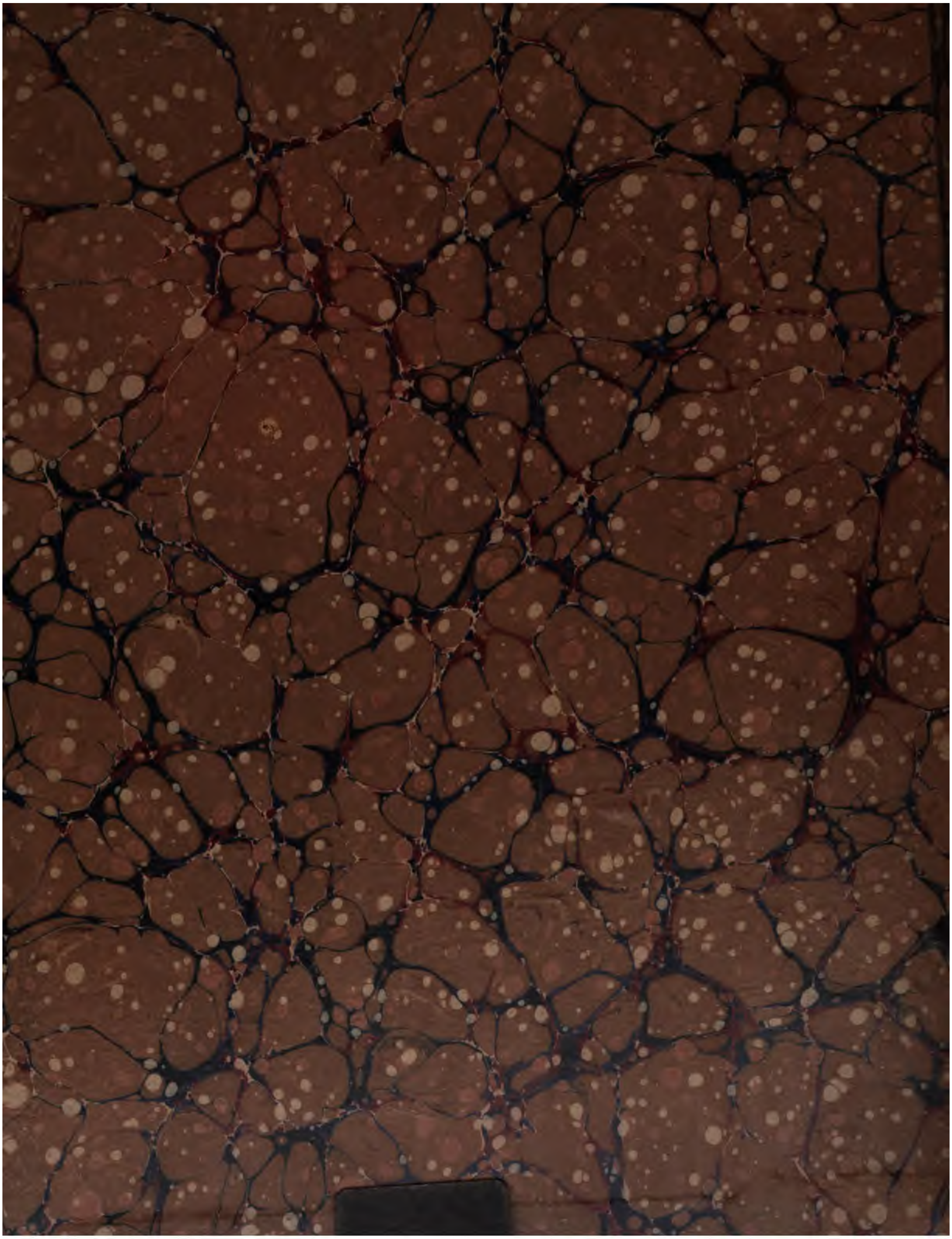

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ATLAS OF ABSORPTION SPECTRA

BY

H. S. UHLER and R. W. WOOD

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INTRODUCTION.

By R. W. Wood.

In spite of the very large amount of work which has been done on absorption spectra, there exists practically no collection of photographed spectra from which one can pick out the media most suitable for any particular line of investigation. The greater part of the published records are drawings made from visual observations, and give no information regarding the optical properties of the media in the ultra-violet. It seems desirable therefore to compile a set of photographic records which are free from the errors liable to enter into observations made by visual methods, and to arrange them in such a way that the medium or media necessary to secure a desired result could be readily found from a mere inspection of the plates.

A great deal of experimental work was necessary before satisfactory photographic records were obtained. The details of the spectrograph and the refinements of the method have been worked out very skilfully by Dr. Uhler, who has done practically all of the experimental work. It was our original plan to include the colored salts of metals, and to examine a large number of colorless substances for peculiarities in the ultra-violet. The solutions of the inorganic compounds could not, however, be investigated in precisely the same manner, owing to their less powerful absorption. Much thicker absorbing wedges were required, and these gave trouble, even when compensated, as a result of dispersion. It was therefore decided to limit the present work chiefly to a study of the aniline dyes, which are used to a much greater extent than the metallic salts, in the preparation of absorbing screens. The absorption spectra of a number of metallic salts, however, have been photographed, as it is believed that many of them will be useful in the preparation of ray filters; some of them are far more transparent in the ultra-violet than the aniline dyes. Even such substances as the salts of erbium, neodymium, and praseodymium are useful in special cases where it is desired to suppress one or more isolated spectral lines. For example, a solution of neodymium has a very narrow and intense band coincident with the D lines, and has therefore the property of cutting out the sodium radiation from a given source, transmitting at the same time nearly the whole of the remainder of the spectrum. The same salt can be used to advantage when working with the new cadmium and zinc arc lamps in quartz tubes, made by Heraeus. The fact that the absorption of each substance in the ultra-violet is recorded, makes the plates of especial value to any engaged in the preparation of screens for spectroscopic or photographic purposes.

For the removal or transmission of one or more isolated lines some other arrangement is often more useful than an absorbing screen. A spectroscope with a slit placed in the focus of the observing telescope (monochromatic illuminator) is frequently all that is necessary. But if more light is required, the following device may be used. A block of quartz from 2 to 4 cm. in thickness, cut perpendicular to the optic axis, is mounted between two Nicol prisms. The transmitted spectrum is crossed by black bands, which result from the rotatory power of the quartz. By adjusting the nicols and varying the thickness of the quartz it is often possible to get rid of the spectrum lines which are not desired, and at the same time to utilize the whole area of the source, which can never be done with the spectroscope. In this way, with a quartz plate 45 mm. thick, the line 4809 of the zinc arc in quartz can be completely removed and the two lines 4721 and 4679 transmitted. This method is especially useful in the study of the fluorescence excited in various bodies by monochromatic light.

If it is necessary to separate radiations of very nearly the same wave-length, for example if we wish to work with the light of one of the two sodium lines, the following arrangement can be used: A quartz plate about 2 cm. in thickness, cut parallel to the axis, is mounted between crossed nicols, with its axis making an angle of 45° with the principal planes of the polarizing prisms. The source is placed behind a vertical slit 2 or 3 mm. in width, and the light, after traversing the polarizing system is brought to a focus by a lens. A number of concentric maxima and minima will be formed, the light of D_1 and D_2 being found in adjacent maxima. The wave-length which is not desired can be stopped by a screen of suitable dimensions placed at the focus of the lens. In this way it is possible to obtain a source of D_1 or D_2 radiation of sufficient intensity to show distinct fringes in a Michelson interferometer. By a curious coincidence this method occurred independently to the writer and to Professor Michelson on the same day. It has been found to give excellent satisfaction. The thickness of the quartz plates used in either of the above cases depends upon the closeness of the lines which it is desired to separate.

We are under great obligation to the Actiengesellschaft für Anilinfabrikation and to Meister, Lucius & Brüning, both of which firms presented the Johns Hopkins University with a large collection of aniline dyes.

ATLAS OF ABSORPTION SPECTRA.

OBJECT OF THE PRESENT INVESTIGATION.

If we look over the literature of the subject of absorption of light we fail to find a collection of absorption spectra presented in such a manner as to enable the observer to select at a glance a substance which produces either general or selective absorption in any specified part of the visible or ultra-violet spectrum. The wave-lengths of the absorption bands and other characteristics of the absorption exhibited by innumerable natural and artificial compounds and mixtures, both inorganic and organic, may be found in a great many books, journals, memoirs, and dissertations. If all of these results were reproduced and catalogued in one volume they would not satisfactorily fulfil the requirements just mentioned, because the different experimenters have had various objects in view and hence they have worked in various and limited parts of the spectrum, have used different numerical dispersions, have employed optical systems of unlike dispersion curves, have not made it possible even to reduce their results to graphical form much less to a common basis of wave-lengths and normal dispersion, etc. The nearest approach to a work of the kind under consideration is made by the publications of J. Formânek, especially the two volumes entitled respectively "Die Qualitative Spektralanalyse anorganischer Körper" and "Spektralanalytischer Nachweis künstlicher organischer Farbstoffe;" Berlin, 1900. Formânek's investigations are very extensive and complete from the point of view explicitly stated in the preface to the last-named volume. It was his aim to develop a practical spectroscopic method of procedure by which any given organic coloring matter could be unambiguously identified. He says:

> **"Das Princip des hier beschreibenen neuen Verfahrens beruht auf der Kombination der spektralanalytischen Beobachtung und der chemischen Untersuchung; dieses Verfahren liefert nicht nur sichere Resultate, sondern sein Vortheil liegt auch darin, dass man mit Hilfe desselben alle einzelnen Farbstoffe von einander unterscheiden kann."**

Formânek, in order to obtain his results, varied the concentrations of his solutions until each absorption band of a given substance became in succession as well defined as possible, so that the wave-lengths of their maxima might be read off with precision. This method is preëminently adapted to locating maxima, but it gives very little, if any, information relative to the absorption between and beyond the maxima, for bodies exhibiting marked selective absorption, and it tells even less about substances presenting weak, general absorption. Another important respect in which Formánek's diagrams fail to give the data required by the first sentence of this section is that he confined his measurements to eye observations, unaided by phosphorescent screens, and hence he omitted the entire ultra-violet region. In fact, his wave-lengths have the limits 420$\mu\mu$ and 741$\mu\mu$, i. e., from "above" the G line to a little "below" the *a* line. Formánek used a prism spectroscope to the dispersion of which he gives no clue.

To fill in this gap in the then existing collections of absorption spectra the present research was begun in the spring of 1903. Its chief object is *to furnish graphical representations, on a normal scale of wave-lengths, of the absorption spectra, both in the visible and in the ultra-violet regions, of a reasonably large number of compounds.*

The most obvious use to which such a collection can be put is the production of color screens either for photographic work or for removing higher orders of spectra from the first order, in the case of diffraction gratings. It also makes possible the selection of such solutions as will transmit relatively narrow, and hence roughly monochromatic, regions of the spectrum. Such solutions are often convenient substitutes for somewhat elaborate pieces of apparatus which first disperse the light by a prism (or grating) and then permit any desired portion of the resulting spectrum alone to continue uninterrupted by means of a suitable slit and screens. Other directions in which the data given below may be of practical value need not be pointed out here.

SELECTION OF MATERIAL, APPARATUS, ETC.

That a great deal of time was consumed in constructing apparatus and in performing preliminary experiments is shown by the fact that, although the investigation was entered upon in the spring of 1903, it was not until July, 1904, that the first really satisfactory negative was obtained. Only aqueous solutions of the aniline dyes have been investigated up to the present time. As is well known, the position of an absorption band may be shifted within wide limits by varying the solvent;* moreover, many aniline dyes are insoluble, or nearly so, in water. On this account it would have been desirable to have made use of the alcohols, benzol, and other organic compounds as solvents for the media under investigation. But difficulties were met with which were not overcome until the study of the dyes was completed. Chief among them may be mentioned the rapid evaporation of the fluid held between the quartz plates. Attempts were made to obviate the difficulty by painting the edges of the wedge with melted paraffin, but the heat of the spark was sufficient to drive off the greater part of the fluid before the exposure was finished. Water is, however, the solvent generally used, and the easiest one to manage. It is moreover free from ultra-violet absorption, which is not true of the majority of the other solvents available, and all dyes which can be dissolved in water can be used for staining gelatin films. The gelatin can be dissolved in the solution of the dye and clean glass plates flowed with the warm solution, or an unexposed photographic plate, after preliminary treatment with thiosulphate of soda and thorough washing, may be stained with the solution of the dye. It is probable that the position of the absorption bands is the same in gelatin as in water, for the indices of refraction of the two media are very nearly the same.

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* *See* Nos. 158 and 165.

ABSORBING MEDIA.

Because of their great variety, strong selective absorption, and general interest, aniline dyes and their related organic compounds were selected as best suited for the study contemplated.

DISPERSING SYSTEM.

In order to obtain reasonably normal spectra a spherical, concave, speculum grating, whose radius of curvature was 98.3 cm., was used. For the first order spectra and for short photographic exposures the astigmatism of the reflector did not produce deleterious effects. This was determined by actual measurements. The length of one line of the ruling was 1.96 cm., and the assemblage of lines covered 5.36 cm. The spectroscopic resolving power was 21,250 (2.125 inches with 10,000 lines per inch). The inconvenience of superposed higher orders will be mentioned later on. To obtain a general idea of the normality of the spectrograms and of the linear dispersion it may be stated that, by calculation one millimeter the center of which was at 214.7$\mu\mu$, or 399.4$\mu\mu$, or 656.3$\mu\mu$, covered 25.77, 25.84, and 25.71 Å. U., respectively, for the spectrum was designed to be normal at the air line 399.4$\mu\mu$.

PHOTOGRAPHIC MATERIAL.

Because of the short radius of curvature of the focal surface (about 49 cm.) celluloid films were employed in most cases. The films used throughout were M. A. Seed's "L-ortho cut negative films," size 5 by 7 inches. The emulsion is by no means equally sensitive over the field of wave-lengths studied, i. e., from 0.2μ to 0.63μ. The chief maximum of sensitiveness is in the yellow, about 0.56μ. A much weaker maximum is near 0.49μ. The middle of the less sensitive intervening region is very roughly 0.52μ.

For the short exposures given throughout, these films are not appreciably influenced by wave-lengths longer than about 0.61μ. The resultant effect of the Nernst glower and the Seed emulsion is best understood by referring to fig. 102, plate 26, for which the times of exposure were, in order, 2 seconds, 5 seconds, 15 seconds, 30 seconds, 1 minute, 2 minutes, and 3 minutes.

Various schemes to make the resultant action more uniform were tried and other makes of films were tested, but no improvement on the simple combination of the Seed emulsion and the Nernst glower resulted, therefore they were used almost exclusively. The Seed films are good in the ultra-violet as is shown by the fact that with an exposure of 5 minutes the aluminium line at 185μμ was clearly recorded. To see if appreciable shifts in the apparent positions of the absorption bands were produced by the yellow maximum and the green minimum of the Seed films, negatives of the same absorbing medium, under exactly the same conditions, were taken on several different makes of films and plates which did not exhibit maxima and minima of sensitiveness for the same wave-lengths. Also, other and independent tests of this possible source of error were made. The conclusion was that no noticeable displacements of the bands were caused. However, in the cases of brown and other visibly colored solutions, exhibiting weak, general absorption, the observer of the appended positives must be careful to distinguish between true absorption and the spurious effects in the vicinity of 0.52μ.

In photographing bands in the orange and red, Cramer "Trichromatic" plates were found to be the best and hence they were used. The plates being plane they had to occupy a mean position with respect to the focal surface of the grating. Since only a comparatively small region of wave-lengths was thus recorded, no measurable errors were introduced. In fact, in the region considered, the second order ultra-violet of a discontinuous spectrum taken on a film and on a plate could be superposed line for line.

The developer used was a simple hydrochinone solution made up according to Jewell's formula.*

SOURCES OF LIGHT.

For wave-lengths from "above" 0.65μ to "below" 0.326μ, and for exposures of about one minute, the Nernst glower was found to be the most satisfactory. Prevailing circumstances made desirable the use of 104 volt glowers on a circuit carrying about 133 cycles. The emissivity of the Nernst lamp varies so very greatly with the e. m. f. impressed upon its terminals that it was obligatory to keep in series with the glower a Thomson A.C. ammeter having a range from zero to two amperes and graduated directly to 0.02 ampere. Fluctuations of more than 0.02 ampere invariably resulted in a spoiled photograph, consequently boxes containing variable metallic resistance were maintained in series with the ammeter and thus, in spite of large changes in the load on the dynamo, due to other experimental circuits, it was possible to prevent the effective current in the filament from changing by more than 0.01 ampere. The current was usually 0.8 ampere or a little less. The ammeter was appreciably more sensitive to small changes in the terminal voltage than a comparably graduated Thomson A. C. voltmeter, because the current shunted through the voltmeter was not negligible in comparison with the current which fed the glower. Among other sources the electric arc was given a fair trial and discarded for two reasons, first, because of the intensity of the carbon and cyanogen bands, and second, because of the inconveniences resulting from its unsteadiness and great emission of heat.

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* L. E. Jewell. Astrophys. Jour., v. XI, 1900, pp. 240–243.

For wave-lengths between the strong ultra-violet of the Nernst glower and 0.2μ a spark discharge in air of about 1 cm. length was used. In obtaining the greater number of the negatives one electrode was composed of an alloy of equal parts by weight of cadmium and zinc and the other was made of sheet brass. The alloy wore away so rapidly that the brass electrode was employed to reduce the labor attendant upon sharpening the terminals. The electrodes were given a form apparently not described before. As is very well known, many spectral lines, both weak and strong, produced by sparks between metallic surfaces extend only a short distance beyond the metal and hence do not offer a continuous source of light across the entire spark gap. In order to obtain a background of uniform intensity from edge to edge of the negatives it was necessary to use some scheme to nullify the effects of the non-uniformity of emission in the spark. One way of accomplishing this is to rapidly translate the electrodes (maintained at a fixed distance apart) back and forth parallel to the length of the slit of the spectrograph by some mechanical device.

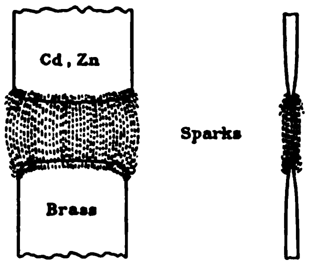

Fig. 1.—Flat and edge views.

The reciprocating action associated with this plan shakes the camera and grating to such an extent as to demand greater rigidity in the apparatus than it usually has. Therefore the electrodes were made in the shape of wedges or chisels with the sharp edges parallel to the slit. The well-known distribution of a rapidly alternating current in a conductor necessitated curving the edges of the electrodes, as is shown in fig. 1, which is $\frac{11}{12}$ natural size. Due to the tearing away of the metal, and to various other causes, the innumerable thread-like sparks changed the positions of their ends so rapidly that the integrating action of the photographic film recorded a perfectly uniform negative for exposures of 15 seconds or more. The exposures generally lasted 75 seconds. The electrodes had to be kept sharp and smooth, for, when this was neglected, the elementary sparks persisted much longer in one position than in another and consequently caused streaks of varying intensity to run along the negatives parallel to their length, as can be seen in some of the positives reproduced in the appended plates, e.g., fig. 99, plate 25.

The current for the spark was obtained in the following manner: An alternating e. m. f. of about 106 volts (133 cycles) was impressed on the terminals of an induction coil of unknown ratio of turns. Eight or nine amperes commonly flowed in the primary. The interrupter of the coil was thrown out of circuit and the coil therefore performed the functions of a transformer. In parallel with the secondary was placed a Leyden jar about 18 inches high and of unmeasured capacity. No auxiliary spark was introduced. The system could spark about 2.5 cm. in air between metallic points.

The great intensity of some of the lines characteristic of all the common metals tried (Al, Cd, Cu, Fe, Pb, Zn, etc.) made these metals undesirable for the present work. Cadmium and zinc were selected only because of the strong continuous background to which they give rise. Uranium, its salts or its earths were not used in this work because they are unmanageable. Naturally the pure metal in air burns to oxide at once; pitchblende can not be worked into a suitable shape (at least, for such specimens as we have been able to obtain); and, pitchblende is so very heterogeneous that the position of the spark can not be depended upon for an instant. To have employed a neutral atmosphere in conjunction with a reciprocating mechanism would have consumed, obviously, too much time and would have demanded too complicated, cumbersome and inconvenient an assemblage of apparatus.

THE CELL.

In order to show the variations in the absorption spectrum of a given substance when the thickness of the absorbing layer changed linearly, a wedge-shaped cell was constructed. Vessels made on this principle have been designed and used often before, notably by Ångström, Gladstone, Govi, Gibbs, Tumlirz, Hodgkinson, F. Melde, Hartley, and others.* Nevertheless, because the precise form of the cell is supposedly new and certainly useful it may not be superfluous to enter into a detailed description of it here. This little piece of apparatus was designed so that the relative positions of the quartz surfaces through which the light entered into, and emerged from, the absorbing liquid could be varied at will, within certain limits. In other words, matters were so arranged that the liquid could be in the form either of a wedge, of variable angle, with zero thickness at the refracting edge, or of a prism of variable angle and finite depth throughout, or of a plane-parallel layer of changeable thickness. To satisfy these conditions it was convenient to rely upon gravitation to preserve certain parts of the cell in mutual contact. This in turn necessitated both the horizontal position of the bottom of the cell and (because it was desirable to reduce the number of reflecting surfaces to a minimum) a vertical type of spectrograph.

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* *See* H. Kayser, "Handbuch der Spectroscopie," v. III, pp. 58, 59.

The cell comprised five separable parts, as follows: (1) A brass framework upon which the other parts rested; (2) a transparent tray, without a lid, which confined the liquid in proper bounds; (3) a transparent boxlike system which gave the upper surface of the liquid the desired position; (4) a vulcanite framework to hold the last mentioned box in place; and (5) four mahogany pins or pegs to fasten the box to its framework.

(1) A side view of this framework is presented in figure 2. There were three micrometer screws, all of the same pitch, viz: 1 turn = $\frac{1}{48}$ in. = 0.053 cm. The heads of the screws were graduated, on their upper surfaces, in ten equal parts. The screw T was in the medial plane of the cell while the remaining screws (T′ only is shown) were at the other end of the system, were equidistant from this plane, and were as far apart as possible. The micrometer screws called for vertical scales on the adjacent brass-work to count whole turns. The handle is denoted by HH. A black fiducial mark, F, on a white ground, enabled the experimenter to tell what position the cell occupied with reference to the length of the slit of the spectrograph. The lower end of F moved over a scale parallel to the slit and in the plane of the jaws of the latter. The flange at the bottom of the framework was made of brass only 0.014 cm. thick so that the absorbing medium might be as near the slit as possible.

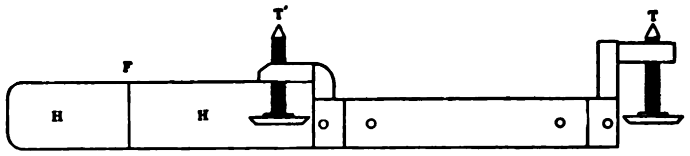

Fig. 2.—Four-fifths natural size.

(2) An accurately ground, plane-parallel plate of quartz 40 mm. long, 18.5 mm. wide, and 2 mm. thick had cemented to its periphery four rectangular sheets of thin glass 8 mm. high. Hence, the greatest depth of liquid which could be studied by the aid of this cell was 6 mm.

(3) In figure 3, *a*, *b*, *c*, and *d* designate the vertices of the section of a quartz plate, made by a plane perpendicular to the plane the trace of which is the line *ad*. *ab* was 2 mm., *ad* was 34.8 mm., and the angle between the planes of *ad* and *bc* was 55 minutes of arc. The horizontal width of the wedge was 10 mm. Glass walls surrounded three sides of the wedge, as the outline indicates. The reason for using the quartz wedge was to counteract the deviation and dispersion produced by the solution in the cell. The angle of the liquid wedge could be varied until the deviation effected by the quartz wedge nullified the average action of the absorbing solution. At first it was supposed that with liquid wedges of 15 or so minutes of arc a plane-parallel quartz plate could be used successfully instead

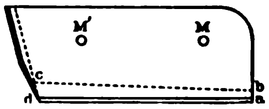

Fig. 3.—Four-fifths nat. size.

of the quartz wedge. This was true for some dyes but for concentrated solutions of certain other dyes (notably the sodium salt of *p*-methoxy-toluene-azo-β-naphthol-di-sulphonic acid) some compensating system was absolutely necessary. Finally, the quartz wedge was made with the utmost care by an expert optician, special pains being taken to have the edge through *d*, perpendicular to the plane *abcd*, as sharply defined as possible, and the surfaces whose traces are denoted by *ad* and *bc* were accurately plane.

(4) Figure 4 presents a side view and an end view of the vulcanite frame into which the quasi-box just described fitted. This frame was shaped out of a single block of vulcanite, for experience showed that a cemented system of several pieces was not durable; also a dielectric was needed to keep the sparks from jumping to the screws. P indicates a little depression which fitted over the point of the screw T. P′ designates the end of a straight line along which the rounded extremity of the screw T′ slid. P″ is the cross-section of a shallow, V-shaped groove along which the pointed end of the third screw, T″, likewise slid. The perforations M, M′, etc., correspond to each other and to the associated wooden pegs mentioned above as (5).

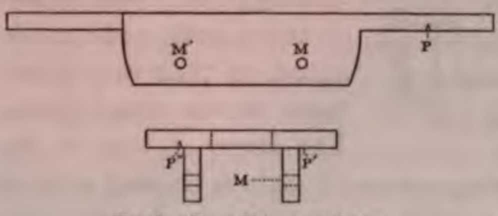

Fig. 4.—Four-fifths natural size.

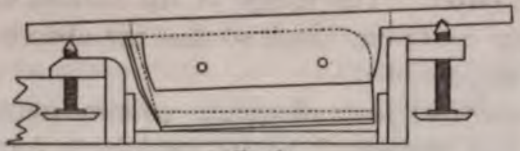

Fig. 5.

Figure 5 is an unconventional sketch of the cell when completely assembled.

A cell of the construction just described is very well suited to the study of thin layers of solutions in solvents of relatively high boiling-points, such as water and amyl alcohol, but, unless inclosed in some suitable vessel, it is not applicable to solvents of lower boiling-points like ethyl alcohol, ether, chloroform, etc.

CEMENTS.

A few words concerning cements may not be superfluous because a great many receipts were tried and none was considered entirely satisfactory. No single cement was found which satisfied the following three necessary conditions: (*a*) Of being unaffected by hot or cold water; (*b*) of being insoluble in the alcohols, ether, chloroform, carbon bisulphide, etc.; (*c*) of drying or setting in three or four days, at most.

The plan used by Prof. H. N. Morse, in waterproofing cells for the study of osmotic pressure, gave the best results and hence it was followed in fastening together the quartz and glass parts of the cell described in the last section. These parts were first fastened together with Khotinsky cement in the usual way, that is, by heating them in an air bath, to any convenient temperature above the melting point of this cement, and by heating a stick of the adhesive mixture in a Bunsen flame and then applying it to the surfaces of the hot quartz and glass.

Since this resinous cement is soluble in ethyl and amyl alcohol and other solvents, and because it is attacked by various liquids, such as an aqueous solution of potassium permanganate, it was necessary to coat the exposed surfaces of the cement with something which was chemically inert towards the solutions to be studied. Such a substance is a solution of rubber in carbon bisulphide. This solution was made and used as follows: From an adequate length of black, soft, rubber connecting-tubing segments about 2 cm. long were cut and heated in an evaporating dish over a Bunsen flame until the sections fused, ran together, and formed a very sticky, viscous liquid. (A single long piece of tubing does not liquefy at all satisfactorily.) The liquid state persisted after the contents of the evaporating dish had been allowed to cool down to about room temperature. Carbon bisulphide was next poured into the dish and the contents of the latter were stirred until a homogeneous solution resulted.

The relative proportions of the carbon bisulphide and rubber used were immaterial and were determined by convenience only The solution can be retained indefinitely in a tightly stoppered bottle and used whenever needed. A thin layer of the solution was painted over the Khotinsky cement, after which the quartz-glass system was heated in an air bath at about 100° C. until the layer became dry and hard, and was no longer sticky. (Of course, during the first part of the process the transparent elements of the cell had to fit over a suitable wooden "form," because Khotinsky cement softens too much at 100° C. to maintain objects in their proper relative positions.) After this another thin coat of the rubber solution was applied and the heating continued. This succession of operations was repeated until a thick, hard, dark-brown covering for the joints was obtained. It then made little difference whether the original cement were present or not, as the hard rubber held the quartz and glass together very satisfactorily.

A cement which dissolves readily in water and in acetic acid but which is not affected by ethyl alcohol, amyl alcohol, carbon bisulphide, glycerin, chloroform, ether, benzol, nitrobenzol, aniline oil B, benzaldehyde, toluol, etc., is made by dissolving 2 pounds of pure gelatin in one quart of water and adding to the resulting solution 7 ounces of nitric acid (sp. gr. 1.35 to 1.42). The final solution is colorless and when applied in thin layers dries in a day or so. It is called Dumoulin's liquid glue. This glue does not keep well, even in a tightly stoppered bottle, and is best made up fresh just before being applied as an adhesive.

Since the completion of the experimental work on the aniline dyes a cell, in the construction of which no cement at all was employed, has been designed and successfully used by one of us.* This cell could retain any liquids which would not attack glass and quartz and, although it was designed to confine the solutions in plane-parallel layers, nevertheless, the principles involved in its construction were such as to admit of extension to the production of a cell which would be wedge-shaped in the interior and would, at the same time, hold organic solvents, prevent evaporation, etc.

THE SPECTROGRAPH.

The essential parts of a vertical section of the spectrograph are outlined in figure 6. They may be tersely described, with the aid of symbols, as follows: In the first place, the elements of the system were adjustable in every respect. Light from the Nernst filament, N, was focused by the concave speculum mirror, R, on the slit, S, whence it continued to the grating, G, from which a portion of it was dispersed in the direction of the sensitized film, F. The distances from the middle of the slit to the centers of the mirror and grating were respectively about 89.5 cm. and 97.1 cm. The electrodes, E, were usually at the distance of 4.2 cm. above the slit and they did not interfere with the passage of the light from the reflector to the slit. No lenses or other reflectors were used. The micrometer head at M indicated the separation of the slit-jaws. Q and Q′ denote a screen system such that when Q was vertical the passage of light from the grating to the camera was not interfered with, whereas when Q was horizontal only ultra-violet light of shorter wave-length than 0.4μ could reach the photographic film. PP is a horizontal platform with a scale along its front edge. By sliding projecting, horizontal, opaque screens of various widths along this platform it was possible to cut out completely any region or regions of wave-lengths desired.

In making certain tests, the platform and sliding screens were very convenient. L is the section of a thin, black, metal shutter capable of motion in a horizontal direction and hence at right angles to the length of the photographic films; in other words, parallel to the slit and to the rulings of the grating. A number of long, rectangular slots or openings, suitably spaced and proportioned, were present in this screen so that strips of different widths of the films or plates could be exposed to the light from the grating without causing any displacement of the sensitized surfaces with reference to the grating and slit. This was necessary for impressing comparison spectra, etc. H and H′ suggest the rack-and-pinion system by the aid of which the films could have unexposed portions brought successively opposite to some selected opening in the slide-screen L. D and D′ denote two of the four doors which gave access to the interior of the spectrograph, and which made it possible to close up the camera light-tight, while making, various adjustments with the rest of the system. The camera was made so that, when it contained neither a film nor a plate, it was possible for the experimenter to look directly at the grating and to make observations with the assistance of an eye-piece.

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*The description of the details of the cell is given on pages 241 to 243 of Publication No. 60 of the Carnegie Institution of Washington, entitled: Hydrates in Aqueous Solution. By Harry C. Jones.

Certain black-on-white scales and ruby-glass windows (Z, for example) enabled the experimenter to know the precise relative positions of the various accessories on the interior of the spectrograph, when the entire system was shut up and exposures were being made. Numerous dull black diaphragms and screens (A_1, A_2, A_3, A_4, A_5, etc.) protected the photographic film from the unusable light which came from the central image, I, and from all the spectra except the one desired. U_1 and O_1 give the extreme rays of so much of the first order spectrum as was studied, that is, U_1 and O_1 correspond respectively to about 0.20μ and 0.625μ. Obviously, the spectrograph was dull black, both inside and out, and contained plaited black velvet in appropriate places. A general idea of the size of the apparatus may be derived from the following dimensions: From R to the plane of BC — 198.5 cm.; BC — 34.5 cm.; the bottom edge perpendicular to BC — 27.5 cm.; BJ — 116 cm.; and JK — 29 cm.

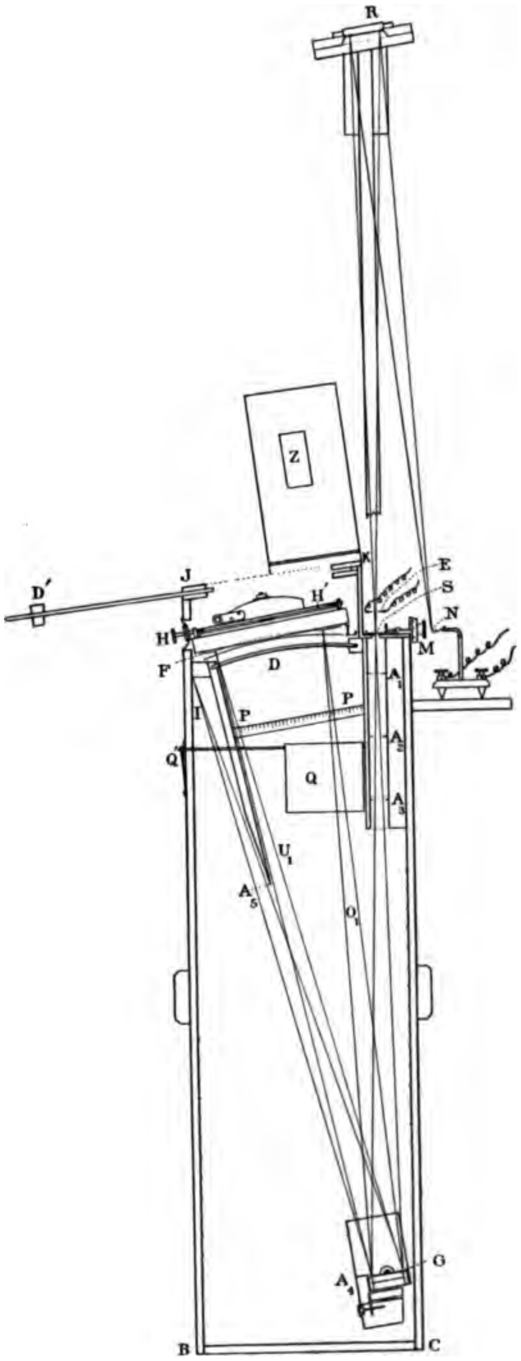

Fig. 6.—One-tenth natural size.

MANNER OF EXPERIMENTING.

SOLUTIONS.

A small, known mass of a selected dye was carefully weighed on a chemical balance, and put at the bottom of a medium-sized test-tube. Then distilled water was run from a burette into the test-tube, and the latter shaken up from time to time, until the resulting solution appeared to have the proper concentration. As would be expected, practice produced skill in judging absorption of visible light, but to get the right concentration with respect to ultra-violet light was not always so easy. The greatest error in measuring the solvents was about 0.2 per cent. Since the concentrations are only intended to serve as general guides to an understanding of the spectrograms, a higher degree of accuracy would have been superfluous. Neither was there any reason, in general, for noting the volume of *solution* which contained a known number of grams of pure solvend; in other words, changes in volume due to the processes of solution were not regarded.

ADJUSTMENT OF THE CELL.

Especial care was taken to remove all coloring matter from the cell before introducing another solution into it. Dust caused more trouble than anything else. After cleaning the quartz and glass elements of the cell the various parts of the latter were assembled and, when a *prism* of liquid was to be studied, the micrometer screws regulated in the following manner: All the screws were turned down so as not to touch the vulcanite framework, and thus to cause the quartz wedge to rest on the quartz plate. Then the screw T had its point elevated again and again until it just touched the deepest part of the depression P. (*See* figures 2, 3, 4, and 5.) This condition was attained by gently rocking the system around the edge *d* of the quartz wedge, somewhat after the fashion of experimenting with certain types of spherometer. Thus the zero position of the cell was determined, before each experiment, of course. Next, guided by the circular and plane scales, the observer turned up the screw T until the desired angle, between the wedge and plate, was known to obtain. After this, the screw corresponding to T′ was turned up until its tip projected far enough into the groove P″ to prevent the quartz wedge and its accessories from sliding over the quartz plate around the point T as pivot, but yet not far enough to raise the vulcanite frame the least bit. Finally, a small amount of the solution was poured into the cell and the latter was then placed on the very thin brass sheet which rested upon and protected the jaws of the slit.

As soon as the cell was placed over the slit and the glower had been lighted the cell was moved forward and backward, parallel to the slit, while one edge of the field of view was examined with an eye-piece, until a position of the cell was obtained for which the light passing through the quartz wedge at its refracting edge (*d* of figure 3) illuminated the very limit of the field of view as seen through the chosen slot of the shutter (L of figure 6). The position of the mark on the handle of the cell (F of figure 2), with respect to the horizontal scale in the plane of the slit-jaws, was then read off. If the cell were then moved, ever so little, in one direction the width of the brightly illuminated field could be seen to be less than the opening in the shutter; whereas, if the cell were translated in the opposite sense no increase in the width of the illuminated field occurred. At this opportunity, eye-observations of the absorption between 0.400μ and 0.625μ were always made and the facts recorded.

When the concentration of the liquid in the cell was much too great or far too small this instrument had to be cleansed and filled with a solution of more suitable absorbing power, obviously, but when the concentration was not too remote from the best value the effective depth of the cell was varied until the desired result was obtained.

All three screws were raised and regulated in an obvious manner when prisms of liquid having nowhere infinitesimal thickness were wanted. When layers of liquid of uniform depth were studied a system much like that shown in figure 3, but which had for bottom a plane-parallel plate of quartz 2 mm. thick, was substituted for the quartz-wedge system.

CALIBRATION OF THE CELL.

The diedral angles formed by the cell were calculated from the dimensions of the instrument, and also from measurements made with a spectrometer.

EXPOSURES AND SPECTROGRAMS.

The majority of the spectrograms consist of three distinct photographs taken side by side and as close together as possible. (*See* the plates.) The width of each photograph was practically the same as the width of the opening in the shutter L. Numerous trials showed that this field of view was completely filled with light, with no overlapping on the grating-side of the opaque portions of the shutter, when the length of the slit was diaphragmed down to 10.5 mm. Consequently, the slit was limited to a length of a very little more than this number and the cell was moved along exactly 10.5 mm. between the taking of two adjacent photographic strips on the same film. By this means, the thickness of absorbing liquid through which the light passed to the very edge of one photographic strip was equal to the thickness subsequently traversed by the light which recorded itself at the contiguous edge of the adjacent strip. Of course, the best appearing records were obtained when the film holder, actuated by the rack-and-pinion system, was moved, by an amount exactly equal to the width of the opening in the shutter. A casual inspection of the positives reproduced in the appended plates shows that mechanical shifts, in wave-lengths, of the strips on one complete spectrogram, with reference to one another, exist. This may mar the appearance of the photographs somewhat, but the ultra-violet spark lines show the magnitude of the displacements so that corrections can be made, and hence the ultimate scientific value of the results is not decreased.

The order of events in taking a complete negative of *three* strips was invariably as follows: The thickest layer of absorbing liquid was over the opening of the slit first, then the intermediate layer, and last of all, the thinnest layer, which usually tapered to infinitesimal depth. This sequence enabled the comparison spectrum to be taken by moving the shutter, L, without jarring the film-holder, so as to minimize the shift of this spectrum relative to the adjacent photographic strip. For negatives of more than three strips precisely the reverse succession was adopted because it was easier to commence with the cell in adjustment and then to raise the quartz wedge parallel to itself than to lower all three micrometer screws by the same number of turns until the quartz wedge just barely came into contact with the quartz bottom of the cell. With the screen Q horizontal the first exposure with the spark was taken. The screen was lowered and the second exposure was made, this time with the Nernst glower. These two exposures produced the first of the three photographic strips. Next the film-holder and cell were moved the proper distances, as explained above. The glower and spark exposures followed in the order named. After again moving the film-holder and cell, the fifth and sixth exposures were produced by the spark and glower respectively. Finally the cell and diaphragm were removed from the slit, another opening in the shutter was adjusted before the film, and the comparison spectrum impressed. In general, the glower exposures lasted 60 seconds, the ultra-violet exposures 75 seconds, and the comparison exposures 35 seconds. The width of the slit was always 0.008 cm. In any one complete spectrogram the exposures to the Nernst light were all equal to each other and those for the ultra-violet were related to one another in the same manner. Experience showed that the intervals 60 and 75 seconds were best suited to cause the overlapping ends of the photographic impressions to blend as if they had been produced simultaneously by light from a single source. With the longest exposures used, the light from the glower did not affect the films and plates for wave-lengths as short as 0.315μ and, since the field photographed did not comprise wave-lengths longer than 0.63μ, there was no trouble produced by the ultra-violet of the second order. The screen Q took care of this matter so far as the spark exposures were concerned. Figures 14 and 15, plate 4, indicate how the processes just explained can be extended to negatives as wide as may be desirable and hence to as deep layers of absorbing liquid as may be wished.*

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*Of course, a cell deeper than 6 mm. would be necessary if the matter were pushed very far.

RESULTS.

INTERPRETATION OF THE CURVES.

If the distances from the edge of a positive which is adjacent to the comparison spectrum (which edge therefore corresponds to zero depth of liquid in the cell) to arbitrary points on the boundary of a sharply-defined absorption curve be called ordinates, and if wave-lengths be considered as abscissæ, we may say that the absorption constants* associated with any two chosen wave-lengths are inversely proportional to the ordinates belonging to these wave-lengths. This statement involves certain assumptions, about emission curves and sensibility curves, a discussion of which will not be given here.

If the edge of an absorption band is a straight line at right angles to the length of the picture it means that the position of this side of the band will not appreciably change with wide variations in the concentration of the solution; in other words, the limit of absorption will remain at the same wave-length regardless of the concentration. This is roughly the case in figs. 4 and 15 of plates 1 and 4 at the respective wave-lengths 0.29μ and 0.515μ, and for most of the narrow bands of figs. 96, 100, and 101. If this condition holds for all the bands of a given substance, which are within or near the confines of the visible spectrum, the color of the light transmitted by the solution will be the same no matter how much the concentration be varied. This is well illustrated by solutions of the salts of neodymium and praseodymium.

When the boundary of an absorption band is a straight line inclined to the axis of wave-lengths it may be inferred that the limit of the band will be displaced in proportion to the change of concentration, and that the factor of proportionality depends upon the angle which the line makes with the axis of abscissae. This is exemplified in fig. 45, plate 12, by the portions of the band, at wave-length 0.47μ corresponding to the thicker layers of liquid.

In like manner, the general relation between the displacements of the limits of absorption and the associated changes in concentration may be easily inferred when the confines of the absorption bands are curved either convex or concave.

EXPLANATION OF THE TABLES.

Two plans suggest themselves for the sequence of the experimental data, viz: (*a*) To classify the material on the basis of the characteristics of the absorption spectra, i. e., the succession, intensity, etc., of the bands and regions of absorption; (*b*) to arrange the results according to the chemical nature of the absorbing media. Because the first method conforms more closely to the professed object of the present research than the second, every scheme consistent with it was tried which suggested itself. The great number of combinations on the negatives of the effects of weak, general absorption with definite, intense bands, combined with more or less uncertainty as to the interpretation of the negatives in the region for which the source of the discontinuous spectrum had to be used, made it impossible to find a satisfactory permutation of the photographic records. Consequently the second plan suggested above was followed as far as the text is concerned. The spectrograms, on the contrary, are arranged, as far as possible, so as not to have widely different absorption spectra succeed one another on the same plate.* The organic coloring matters succeed one another in the same order as is given to them in the English translation by A. G. Green of a book by G. Schultz and P. Julius entitled "A Systematic Survey of the Organic Colouring Matters" (Macmillan & Co., London, 1904). This connection between the contents of the volume just named and the material recorded below has the advantage of making it easy to find out many things about the dyes which can not be appropriately given here, such as the names of their discoverers, their literature, patents, methods of preparation, their behavior with various reagents, chemical constitution, etc.

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*κ_λ in $I'_\lambda = I_\lambda e^{-\kappa_\lambda l}$

The descriptive tables following this explanatory section present the experimental results in the following order:

(1) The absorption of a small number of interesting intermediate products, so-called, arranged according to the alphabetical order of their names.

(2) The absorption of such dyes as were studied and were capable of identification with the dyes discussed in the book by Schultz & Julius.

(3) The absorption of such dyes as were not unquestionably the same as any given in the reference volume. The accounts of these dyes follow the alphabetical order of their *commercial* names.

(4) The absorption of certain miscellaneous objects of more or less interest, in alphabetical order.

Whenever a number without qualification is given to a substance it refers to the present account, but when a number is quoted from the volume by Schultz & Julius attention is called to the fact by the abbreviation S. & J.

In the brief account of any one dye the details are presented in the sequence explained by the following sentences:

First. The arbitrary number of the substance in the present list is given.

Second. The commercial name of each substance is recorded precisely as it was labeled by the firm which furnished the coloring matter. When two firms sent the same dye under the same or under different names the circumstance is explicitly presented.

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*Plate 2 is an exception to this statement.

Third. Immediately after the commercial name that of the factory is given. The dyes were obtained from three sources. Both the Actiengesellschaft für Anilinfabrikation and Meister, Lucius & Brüning presented a large number of dyes of their manufacture to the Johns Hopkins University. The other dyes were purchased from the firm of Eimer & Amend, New York.

The following abbreviations are used throughout.

[A.] Actiengesellschaft für Anilinfabrikation, Berlin (The Berlin Aniline Co.).
[A. A.C.] The Albany Aniline Color Works, Albany, New York.
[B.] Badische Anilin-und Sodafabrik, Ludwigshafen am Rhein (The Baden Co.).
[By.] Farbenfabriken vorm. Fr. Bayer & Co., Elberfeld (The Bayer Co.).
[C.] Leopold Cassella & Co., Frankfurt am Main.
[D.] Dahl & Co., Barmen.
[D. H.] L. Durand, Huguenin & Co., Basle and Hüningen.
[G.] J. R. Geigy, Basle.
[I.] Société pour l'Industrie Chimique (formerly Bindschedler & Busch), Basle.
[K.] Kalle & Co., Biebrich am Rhein.
[M.] Farbwerke vorm. Meister, Lucius & Brüning, Höchst am Main (Meister, Lucius & Brüning, Limited).
[O.] K. Oehler, Offenbach am Main.
[P.] Société Anonyme des Matières Colorantes de St. Denis, Paris.

Fourth. The chemical name of the absorbing medium is given.

Fifth. Reference is made either to the figure (or figures) and plate which belong to the substance under discussion itself or to a figure which is very much like the spectrograms of the dye considered.

Sixth. When possible, the number of the dye or the page of the intermediate product, as found in the volume of Schultz & Julius, is recorded.

Seventh. The color and superficial character of the dry coloring matter is suggested.

Eighth. The color of the solution as observed in a test-tube is followed by the color in the cell. The change of color with thickness is often significant.

Ninth. Then follows the concentration in grams of dry solvend in a liter of solvent. The term "saturated" is to be understood in its general, practical sense and not in the almost unattainable, theoretical sense. Parenthetical, qualifying words, such as "(heated, filtered)," call attention to the fact that the substance does not dissolve readily in water, or that the solution contained gritty, foreign material, etc.

Tenth. Next is given the angle between the quartz plates forming the top and bottom of the various cells used. In the same line the numbers denote in order the minimum and maximum depths of solution through which the light passed before acting upon the outer limits of the negative. The intermediate thicknesses vary linearly, of course. The same angle is not always associated with the same maximum depth, even when the minimum thickness is unchanged, because several cells of different dimensions were employed.

Eleventh. Finally, a brief account of the most noticeable characteristics of the absorption spectrum, between the limits 0.20μ and 0.63μ, is furnished.

The results of eye-observations of the absorption spectra come first and serve as checks on the photographic records. The data obtained visually are qualitatively reliable for all strong bands between 0.40μ and 0.63μ. For cases of very weak, general absorption much less importance must be ascribed to the visual results because, unfortunately, the cells were not constructed so as to present side by side, in the field of view, two spectra, the one of the light after passing through the absorbing solution, the other of the unabsorbed light direct from the Nernst glower.

When the solution is fluorescent, or decomposes when ultra-violet light falls upon it, or possesses a characteristic odor, etc., the facts are noted. That the spectrograms are not distorted by the presence of fluorescent light, but give as true records of the absorption spectra of fluorescent compounds as they do for non-fluorescent solutions, was ascertained by direct experiments. (In particular, see the record for solution No. 107.)

Lastly, the *approximate* wave-lengths of the maxima and minima of absorption, as obtained from the spectrograms, are given, beginning near 0.20μ and continuing to 0.63μ. When the wave-lengths of the "ends" of a region of absorption are given they obviously have significance only under the conditions of thickness of absorbing layer, of concentration, of length of photographic exposure, etc., which prevailed at the time when the spectrogram was taken. The maxima are not subject to the same limitations. The fact that the Seed films can produce spurious absorption bands in the green must be again emphasized. (*See* figure 102, plate 26.)

When the end of the spectrogram, which marks the fading away of the sensitiveness of the emulsion from the yellow to the orange, is practically a straight line perpendicular to the length of the spectrogram it means that there is no appreciable general absorption in this locality, but when the limit just specified is approximately a right line inclined at an obtuse angle to the positive direction of the axis of wave-lengths it signifies that appreciable general absorption is present in this region.

TABULATED DATA OF ABSORPTION.

INTERMEDIATE PRODUCTS.

1. Amidonaphtholdisulphonic Acid H. (M.)

Fig. 1, pl. 1; pp. 57 and 58, S. & J.

Grayish-white lumps. In solution brownish yellow, colorless.

Saturated.

Angle 27.3′. Depth 0 to 0.25 mm.

No visible absorption. Intense blue fluorescence. Ultra-violet absorption ends about 0.347μ.

2. β-Naphtholdisulphonic Acid G. (M.)

Figs. 2 and 5, pl. 1; p. 51, S. & J.

Pinkish-white powder. In solution colorless.

Saturated.

Angle 27.3′. Depth 0 to 0.25 mm.

No visible absorption. Intense blue fluorescence. Absorption ends very definitely and follows approximately a straight line from 0.346μ to 0.356μ. Fig. 5 shows absorption exhibited by a solution made by diluting a certain volume of the saturated solution to eight times its original value.

3. *p*-Nitraniline. (Powder, "extra.") (M.)

Page 12, S. & J.

Lemon-yellow powder. In solution yellow, faint yellow.

Saturated.

Angle 37.1′. Depth 0 to 0.34 mm.

No visible absorption is produced by a column 6 cm. deep. Entire ultra-violet absorption is weak. A region of slight absorption from 0.20μ to 0.255μ is followed by transparency as far as 0.34μ. Faint absorption extends from 0.34μ to 0.40μ. From 0.40μ to 0.63μ no absorption is noticeable.

4. *o*-Nitrobenzaldehyde. (M.)

Page 61, S. & J.

White needles. In solution colorless.

Saturated.

Angle 31.2′. Depth 0 to 0.29 mm.

Extremely weak absorption from 0.20μ to 0.24μ. Transparent from 0.24μ to 0.63μ.

5. *p*-Nitrosodimethylaniline.

Fig. 3, pl. 1; p. 32, S. & J.

Dark-green, crystalline powder. In solution brownish yellow, clear yellow.

Saturated.

Angle 23.4′. Depth 0 to 0.21 mm.

Strong absorption in violet and blue increasing towards the ultra-violet. A remarkably transparent region extends from 0.30μ to 0.375μ. All the strong lines between 0.324μ and 0.363μ are transmitted with almost no decrease in intensity.[*] A very round band stretches from 0.375μ to 0.448μ with its maximum at 0.432μ. Complete transparency from 0.49μ to 0.63μ.

6. Resorcine (techn. pure). (M.)

Fig. 4, pl. 1; p. 45, S. & J.

White, crystalline lumps. In solution colorless.

Nearly saturated.

Angle 29.3′. Depth 0 to 0.27 mm.

No visible absorption. Very faint yellow in a layer a decimeter thick. Absorption ends very abruptly and shows an almost vertical right line determined by 0.287μ and 0.293μ.

COLORING MATTERS.

7. Naphthol Yellow. (A.) Naphthol Yellow S. (M.) Sodium salt of dinitro-α-naphthol-β-monosulphonic acid.

Fig. 42, pl. 11; No. 4, S. & J.

Orange-yellow powder. In solution brownish yellow, pure yellow.

Saturated (heated).

Angle 31.2′. Depth 0 to 0.29 mm.

Intense band in violet, ultra-violet side invisible. Absorption decreases from 0.20μ towards 0.335μ. Transparent region around 0.335μ. A pair of overlapping bands extends from about 0.345μ to 0.465μ. Their maximum absorption is at 0.385μ and their least absorption is at 0.41μ. Very transparent from 0.465μ to beyond 0.63μ.

8. Aurantia. Ammonium salt of hexanitrodiphenylamine.

Fig. 39, pl. 10; No. 6, S. & J.

Reddish-brown crystals. In solution dull red, yellow.

10 g. per liter (filtered).

Angle 42.5′. Depth 0 to 0.36 mm.

General absorption in violet. Absorption decreases from 0.20μ towards 0.28μ. Transparent region from 0.28μ to 0.33μ. Wide band from 0.33μ to 0.49μ with its maximum

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[*] R. W. Wood. "On Screens Transparent only to Ultra-Violet Light and their use in Spectrum Photography." Phil. Mag., v. 5, Feb., 1903, pp. 257-263.

8. Aurantia—*Continued.*

at 0.41μ. As the concentration increases the absorption encroaches much faster on the transparent region in the ultra-violet than on the limit in the yellow. Very transparent to yellow and orange.

9. Fast Yellow. (B.) Sodium salt of amidoazotoluene-disulphonic acid.

Somewhat similar to fig. 40, pl. 10; No. 9, S. & J.

Brownish-yellow powder. In solution brownish yellow, yellow.

15 g. per liter.

Angle 23.4′. Depth 0 to 0.21 mm.

Absorption in violet and blue. The region of partial transmission in the ultra-violet is not as complete for solution No. 9 as for solution No. 32. Also the boundaries of the violet band are somewhat more definite for the former solution than for the latter. The less refrangible side of this band is more like the corresponding region for solution No. 129, fig. 13, pl. 3. Absorption decreases gradually from 0.20μ to semi-transparency at about 0.34μ. A wide, diffuse band extends from this region to about 0.475μ. Its maximum is at 0.40μ. Transparent from 0.475μ to 0.63μ.

10. Orange G. (A.) Sodium salt of benzene-azo-β-naphthol-disulphonic acid G.

Fig. 30, pl. 8; No. 14, S. & J.

Yellowish-red powder. In solution red, yellow.

Saturated (heated).

Angle 21.3′. Depth 0 to 0.18 mm.

Strong absorption in blue and green. Sharp on yellow edge. Two ultra-violet bands meet at about 0.29μ in a semi-transparent spot. The maximum of the less refrangible band is 0.325μ. This strong band meets a very weak one at 0.365μ. The center of the weak band is 0.39μ. The weak band joins an intense one at 0.42μ. This last band joins a still stronger band, from which it is not resolved, at 0.485μ. The maximum of the stronger band is at 0.505μ. Absorption ceases at 0.53μ. Complete transparency to 0.63μ.

11. Ponceau 2 G. (M.) Sodium salt of benzene-azo-β-naphthol-disulphonic acid R.

Fig. 6, pl. 2; No. 15, S. & J.

Bright-red powder. In solution yellowish red, yellow.

7 g. per liter (filtered).

Angle 27.3′. Depth 0 to 0.25 mm.

Comparatively weak band in the blue-green, with a shadowy, fainter companion on the yellow side. Absorption decreases gradually from 0.20μ to 0.34μ. The nearly transparent region from 0.34μ to 0.44μ is interrupted by a very faint band having its maximum at 0.39μ. The pair of stronger bands extends from 0.44μ to 0.545μ. Transparent from 0.545μ to 0.63μ. Same empirical formula as No. 10. No. 10 is derived from the G acid, while No. 11 is a salt of the R acid.

12. Chrysoïdine. Hydrochloride of diamidoazobenzene.

Fig. 7, pl. 2; No. 17, S. & J.

Reddish-brown powder. In solution brown, yellow.

10 g. per liter (filtered).

Angle 23.4′. Depth 0 to 0.21 mm.

Absorption in violet, blue, and green with maximum in the indigo. Absorption decreases from 0.20μ to 0.33μ. Transparent from 0.33μ to 0.36μ. A pair of broad, unseparated bands absorbs from 0.36μ to 0.54μ. The band of greater refrangibility is the more intense and has its maximum at 0.43μ. Transparent from 0.54μ to 0.63μ. The less refrangible band disappears first on dilution. A five-strip negative shows that the outer boundaries of the pair of bands are steep and definite.

13. Chromotrope 6 B. (M.) Sodium salt of *p*-acetamidobenzene-azo-1:8-dioxynaphthalene disulphonic acid.

Fig. 8, pl. 2; No. 38, S. & J.

Grayish-brown powder. In solution red, pink.

5.71 g. per liter.

Angle 11.7′. Depth 0 to 0.11 mm.

Strong absorption in green-yellow. Transparent from 0.35μ to 0.465μ. A strong band has its beginning at 0.465μ and its maximum at 0.515μ. The less refrangible side joins a weak companion band extending into the orange and red. More dilute solutions show that the intense band is symmetrical with respect to its max-

13. Chromotrope 6 B—*Continued.*
imum until it joins the associated band. More concentrated solutions show very distinctly the weaker band in the orange-red.

14. Azo Coccine 2 R. (A.) Sodium salt of xylene-azo-α-naphthol-*p*-sulphonic acid.
Fig. 9, pl. 2; No. 50, S. & J.
Reddish-brown powder. In solution salmon pink, salmon pink.
Saturated (heated).
Angle 27.3′. Depth 0 to 0.25 mm.
Narrow band in the blue-green. An absorption band of very indefinite edges extends from about 0.48μ to 0.53μ with its maximum at 0.505μ. Transparent from 0.53μ to 0.63μ.

15. Brilliant Orange G. (M.) Sodium salt of xylene-azo-β-naphthol-mono-sulphonic acid.
Fig. 31, pl. 8; No. 54, S. & J.
Cinnabar-red powder. In solution yellowish red, deep yellow.
7 g. per liter.
Angle 23.4′. Depth 0 to 0.21 mm.
Intense absorption in blue-green and blue. Very sharp on the yellow side. Absorption decreases from 0.20μ to weak absorption at 0.295μ, then increases to maximum absorption at 0.32μ. At 0.355μ semi-transparency obtains. A definite band has its maximum at 0.395μ and joins the next band at 0.43μ. The next band has its maximum at 0.48μ and joins the adjacent band at 0.505μ. The final band has a maximum at 0.52μ. Absorption ends at 0.545μ. Complete transparency to 0.63μ. The band at 0.395μ disappears rapidly with dilution. Same empirical formula as solution No. 14.

16. Ponceau 2 R. (A.), (M.) Sodium salt of xylene-azo-β-naphthol-disulphonic acid.
Similar to fig 55, pl. 14; No. 55, S. & J.
Brownish-red powder. In solution red, pink.
5 g. per liter (heated).
Angle 27.3′. Depth 0 to 0.25 mm.
Hazy-edged band in the blue-green. Similar absorption to that of solution No. 17 in the ultra-violet and identical with it in the visible region.

17. Ponceau 3 R. (A.), (M.) Sodium salt of ψ-cumene-azo-β-naphthol-disulphonic acid.

17. Ponceau 3 R—*Continued.*
Fig. 55, pl. 14; No. 56, S. & J.
Dark-red powder. In solution red, pink.
5 g. per liter (heated).
Angle 29.3′. Depth 0 to 0.27 mm.
An absorption band is in the blue-green. It has its maximum at 0.50μ and extends from about 0.47μ to 0.54μ. Transparent from 0.54μ to 0.63μ.

18. Crystal Ponceau 6 R. (A.), (M.) Sodium salt of α-naphthalene-azo-β-naphthol-disulphonic acid.
Similar to fig. 55, pl. 14; No. 64, S. & J.
Brownish-red crystals with golden reflex. In solution light red, pink.
5 g. per liter (heated).
Angle 27.3′. Depth 0 to 0.25 mm.
Hazy-edged band in the blue-green and green. Similar absorption to that of solution No. 17. The ultra-violet absorption, however, is somewhat more intense and extends to greater wave-lengths for solution No. 18 than for solution No. 17. The visible band extends from 0.465μ to 0.56μ with its maximum at 0.51μ.

19. Bordeaux B. (M.) Sodium salt of α-naphthalene-azo-β-naphthol-disulphonic acid.
Similar to fig. 19, pl. 5; No. 65, S. & J.
Brown powder. In solution red, red.
4.18 g. per liter.
Angle 42.5′. Depth 0 to 0.36 mm.
Hazy-edged band in the green. The sides of the band in the green and the orange end of the spectrogram slope a little more for solution No. 19 than for No. 106. Band from 0.485μ to 0.545μ, with maximum at 0.515μ. The least refrangible ends for all the spectrograms slope, thus showing that there is some general absorption in the orange. More concentrated solutions show that the greatest transparency occurs at 0.414μ. Same empirical formula as No. 18.

20. Coccinine B. (M.) Sodium salt of *p*-methoxy-toluene-azo-β-naphthol-disulphonic acid.
Similar to fig. 55, pl. 14; No. 73, S. & J.
Dark-red powder. In solution bright-red, red.
13.64 g. per liter.
Angle 12.8′. Depth 0 to 0.11 mm.
Strong absorption in green-yellow. Similar to solution No. 17, save that

20. Coccinine B—*Continued.*
a weak absorption band seems to have the limits 0.315μ and 0.355μ, with a maximum at 0.33μ. Intense band from 0.465μ to 0.555μ, with a maximum at 0.510μ. Transparent from 0.555μ to 0.63μ. Very concentrated solutions or deeper layers show that the transparent region on both sides of 0.41μ becomes opaque much faster than the orange and red region. Red is transmitted when all shorter wave-lengths are absorbed completely. The solution exhibits strong dispersive power.

21. Eosamine B. (A.) Sodium salt of *p*-cresol-methyl-ether-azo-α-naphthol-disulphonic acid.
Fig. 52, pl. 13; No. 74, S. & J.
Reddish-brown powder. In solution yellowish red, pink.
8.89 g. per liter.
Angle 21.3′. Depth 0 to 0.18 mm.
Strong band in blue-green and green. Intense, round band from 0.465μ to 0.565μ, with its maximum at about 0.52μ. Transparent from 0.565μ to 0.63μ. Same empirical formula as No. 20.

22. Erika B. (A.) Sodium salt of methylbenzenyl-amido-thio-xylenol-azo-α-naphthol-disulphonic acid.
Fig. 57, pl. 15; No. 78, S. & J.
Reddish-brown powder. In solution red, pink.
6.67 g. per liter.
Angle 19.5′. Depth 0 to 0.18 mm.
Strong absorption in blue, green, and green-yellow. Two unresolved bands absorb strongly from 0.46μ to 0.59μ. The more refrangible band shows greater intensity than its companion and has its maximum at 0.52μ. Slight absorption in the orange is followed by greater transparency in the red.

23. Emin Red. (A.) Sodium salt of methylbenzenyl-amido-thioxylenol-azo-β-naphthol-sulphonic acid.
Fig. 29, pl. 8; No. 80, S. & J.
Red powder. In solution red, pink.
Saturated (heated).
Angle 31.2′. Depth 0 to 0.29 mm.
Weak, hazy absorption in blue and green. Strong absorption from 0.20μ to about 0.35μ, then a rather rapid decrease in absorption sets in. From 0.38μ to 0.45μ a semi-transparent region exists. A round band extends from 0.45μ to 0.54μ. Its maximum is near 0.495μ. The less refrangible side of this band is far more definite than its ultra-violet edge. Transparent from 0.54μ to 0.63μ.

24. Janus Green. (M.) Chloride of safranine-azo-dimethylaniline.
No. 81, S. & J.
Olive-green, crystalline powder. In solution blue, blue.
4.6 g. per liter.
Angle 17.0′. Depth 0 to 0.14 mm.
Band in orange and orange-red. Transparent to pure red. Very general absorption in ultra-violet, decreasing gradually from 0.20μ to 0.40μ. Transparent from 0.40μ to 0.515μ. The absorption band begins at 0.515μ.

25. Tropæoline O. (C.) Sodium salt of *p*-sulphobenzene-azo-resorcinol.
Similar to fig. 37, pl. 10; No. 84, S. & J.
Brown powder. In solution wine-color, yellow.
Saturated (heated).
Angle 25.5′. Depth 0 to 0.21 mm.
Faint absorption in the violet. Similar absorption to that of solution No. 81. Weak absorption from 0.20μ to 0.275μ. Transparent from 0.275μ to 0.325μ. A weak, hazy band extends from 0.325μ to 0.41μ with its maximum around 0.37μ. Transparent from 0.41μ to 0.63μ.

26. Tropæoline OOO No. 1. Sodium salt of *p*-sulphobenzene-azo-α-naphthol.
Similar to fig. 31, pl. 8; No. 85, S. & J.
Reddish-brown powder. In solution red, salmon pink.
6.67 g. per liter.
Angle 21.3′. Depth 0 to 0.18 mm.
Absorption in violet, blue, and green. Similar absorption to that of solution No. 15. Rather strong absorption continues from 0.20μ to about 0.33μ and then decreases rapidly to semi-transparency. A tolerably transparent region is from 0.35μ to 0.37μ. Three unresolved bands with maxima at about 0.41μ, 0.48μ, and 0.52μ follow. The intermediate points of less intensity of absorption are 0.435μ and 0.450μ. At 0.545μ the absorption ceases. Transparent from 0.545μ to 0.63μ.

27. Tropæoline OOO No. 2. Sodium salt of *p*-sulphobenzene-azo-β-naphthol.
No. 86, S. & J.
Bright, orange powder. In solution deep red, salmon pink.
14 g. per liter.
Angle 21.3′. Depth 0 to 0.18 mm.
Visible absorption and spectrogram identical with No. 26. Similar absorption to that of solution No. 15. Nos. 26 and 27 have the same empirical formulæ, but differ by α and β in the naphthol.

28. Methyl Orange III. (P.) Sodium salt of *p*-sulphobenzene-azo-dimethylaniline.
Fig. 41, pl. 11; No. 87, S. & J.
Ocher-yellow powder. In solution red, yellow.
Saturated (heated).
Angle 27.3′. Depth 0 to 0.25 mm.
Absorption in blue and green. A strong band extends from 0.36μ to 0.525μ. This band is very round with its maximum at 0.44μ. Transparent from 0.525μ to 0.63μ.

29. Tropæoline OO. (C.) Sodium salt of *p*-sulphobenzene-azo-diphenylamine.
Similar to fig. 40, pl. 10; No. 88, S. & J.
Yellow powder. In solution yellowish red, yellow.
6 g. per liter (heated and filtered).
Angle 25.5′. Depth 0 to 0.21 mm.
Delicate absorption in violet and blue. Similar absorption to that of solution No. 32. The extreme ultraviolet absorption is weak because the lines near 0.23μ show on all three photographic strips. From 0.385μ to 0.47μ a weak absorption band obtains with its maximum at 0.43μ. The substance is very transparent to yellow and red. Nos. 29 and 32 have the same empirical formulæ. No. 29 is the para-compound and No. 32 is the meta-. No. 29 shows weaker absorption than No. 32.

30. Curcumeïne. (A.) Mixture of nitrated diphenylamine yellow with nitrodiphenylamine.
Fig. 12, pl. 3; No. 91, S. & J.
Ocher-yellow powder. In solution red, yellow.
Saturated (heated).
Angle 27.3′. Depth 0 to 0.25 mm.

30. Curcumeïne—*Continued.*
Absorption in violet, blue, and blue-green. Absorption complete at 0.20μ, decreasing very gradually with comparatively definite contour to 0.455μ. Transparent from 0.455μ to 0.63μ.

31. Azo Acid Yellow. (A.) Azo Yellow, concentrated. (M.) Mixture of nitrated diphenylamine yellow with nitro-diphenylamine.
Similar to fig. 12, pl. 3; No. 92, S. & J.
Ocher-yellow powder. In solution brownish yellow, yellow.
Saturated (heated).
Angle 27.3′. Depth 0 to 0.25 mm.
Strong absorption of violet, blue, and blue-green. Similar absorption to that of solution No. 30. Absorption is nearly complete and uniform from 0.20μ to about 0.39μ. Then the absorption decreases in a gently sloping curve to about 0.505μ. Transparent to yellow and red. Nos. 30 and 31 are mixtures of the same constituents and have very similar regions of absorption.

32. Metanil Yellow. (A.) Sodium salt of *m*-sulphobenzene-azo-diphenylamine.
Fig. 40, pl. 10; No. 95, S. & J.
Brownish-yellow powder. In solution yellowish red, yellow.
4.29 g. per liter (filtered).
Angle 23.4′. Depth 0 to 0.21 mm.
Absorption in violet and blue. A band with very indefinite boundary extends from about 0.36μ to 0.47μ. The maximum is near 0.41μ. Transparent to yellow and red. A very concentrated solution shows complete absorption from 0.20μ to 0.51μ with a semi-transparent spot at 0.34μ and maximum absorption at 0.40μ. Absorption ceases abruptly at 0.535μ.

33. Naphthylamine Brown. Sodium salt of *p*-sulphonaphthalene-azo-α-naphthol.
Similar to fig. 11, pl. 3; No. 101, S. & J.
Brown powder. In solution reddish brown, almost colorless.
11.11 g. per liter (heated and filtered).
Angle 30.0′. Depth 0 to 0.45 mm.
Very weak, general, indefinite absorption for all visible colors of shorter wave-lengths than the yellow. Similar absorption to that of solution No. 47. Absorption was nearly complete from 0.20μ to 0.274μ. From

33. Naphthylamine Brown—*Continued.*
the latter wave-length the absorption decreased very gradually to a maximum of semi-transparency at about 0.43μ. The apparent absorption at 0.52μ is much exaggerated by the lack of relative sensitiveness of the photographic film at this spot. Very slight absorption from 0.55μ to 0.63μ. A weaker solution presented only ultra-violet absorption.

34. Fast Red A. (A.) New Coccine O. (M.) Sodium salt of *p*-sulphonaphthalene-azo-β-naphthol.
Fig. 27, pl. 7; No. 102, S. & J.
Brownish-red powder. In solution red, pink.
5 g. per liter (heated).
Angle 27.3'. Depth 0 to 0.25 mm.
Hazy absorption in blue-green and general absorption in blue. Muddy-looking solution. Two partially resolved bands extend from 0.415μ to 0.54μ with maxima of absorption at about 0.45μ and 0.505μ. The less refrangible band is the more intense. Orange and red are transmitted, but the sloping end of the photograph shows that slight, general absorption is present in orange. Nos. 33 and 34 have the same empirical formulæ. They differ by α and β naphthol.

35. Azo Rubine S. (A.) Sodium salt of *p*-sulphonaphthalene-azo-α-naphthol-*p*-sulphonic acid.
Similar to fig. 55, pl. 14; No. 103, S. & J.
Brown powder. In solution red, pink.
10 g. per liter.
Angle 19.5'. Depth 0 to 0.18 mm.
Absorption in green. Much like solution No. 17 with slight differences in the ultra-violet. Absorption decreases from 0.20μ to 0.27μ. The strong lines at 0.255μ and 0.275μ are transmitted by the deepest layer. Absorption increases from 0.27μ to a maximum at 0.315μ. Then the absorption decreases to approximate transparency at 0.36μ. Transparent from 0.36μ to 0.465μ. Strong band from 0.465μ to 0.555μ with maximum at 0.51μ. The visible band is in the same place as the like band of No. 20, but the ultra-violet is different. Transparent to orange and red.

36. Fast Red, extra. (A.) Sodium salt of *p*-sulphonaphthalene-azo-β-naphthol-monosulphonic acid.
Similar to fig. 55, pl. 14; No. 105, S. & J.
Reddish-brown powder. In solution red, pink.
7 g. per liter.
Angle 27.3'. Depth 0 to 0.25 mm.
Absorption in blue-green and green. Absorption the same throughout as for No. 35 except the position of the visible band. No. 36 absorbs from 0.460μ to 0.545μ with the maximum at 0.505μ. Same empirical formula as for No. 35.

37. New Coccine. (A.) Sodium salt of *p*-sulphonaphthalene-azo-β-naphthol-disulphonic acid.
Similar to fig. 52, pl. 13; No. 106, S. & J.
Scarlet-red powder. In solution yellowish red, pink.
10 g. per liter.
Angle 23.4'. Depth 0 to 0.21 mm.
Absorption in blue-green and green. Similar absorption to that of solution No. 21. The ultra-violet absorption of solution No. 37 seems to consist only of one band whereas that of solution No. 21 seems to be separated into two bands by a minimum of absorption near 0.274μ. Absorption decreases from 0.20μ to transparency about 0.37μ. A strong, round band from 0.445μ to 0.56μ has its maximum at 0.51μ. Transparent from 0.56μ to 0.63μ.

38. Fast Brown 3 B. (A.) Sodium salt of sulphonaphthalene-azo-α-naphthol.
Similar to fig. 23, pl. 6; No. 111, S. & J.
Dark-brown, glistening powder. In solution reddish brown, faint brown.
15 g. per liter.
Angle 27.3'. Depth 0 to 0.25 mm.
Absorption most intense in blue-green and green with slight general absorption on both sides. Similar absorption to that of solution No. 60. A fairly strong band from 0.46μ to 0.54μ has its maximum at 0.51μ. No definite band from 0.54μ to 0.63μ, but general absorption is made evident by the slope of the end of the negative.

39. Mordant Yellow O. (M.) Sodium salt of sulphonaphthalene-azo-salicylic acid.
Similar to fig. 13, pl. 3; No. 116, S. & J.
Yellow powder. In solution reddish yellow, yellow.
10 g. per liter.
Angle 31.2′. Depth 0 to 0.29 mm.
Absorption in the violet only. Similar absorption to that of solution No. 129. Strong absorption from 0.20μ to 0.28μ. Slight weakening of absorption from 0.28μ to 0.34μ. Absorption attains a maximum at 0.36μ and then slopes gradually, with a comparatively definite edge, to transparency at 0.44μ. From this point to 0.63μ complete transparency exists.

40. Dianil Yellow R. (M.)
Similar to fig. 37, pl. 10; No. 124, S. & J.
Orange-yellow powder. In solution clear yellow, yellow.
Saturated.
Angle 31.2′. Depth 0 to 0.29 mm.
Faint absorption in violet. Similar absorption to that of solution No. 81. Absorption is comparatively strong at 0.20μ and decreases to partial transparency near 0.295μ. A tolerably weak band extends from this region to about 0.435μ. Its maximum is indeterminate. Transparent from 0.44μ to 0.63μ.

41. Resorcine Brown. (A.) Sodium salt of xylene-azo-resorcin-azo-benzene-*p*-sulphonic acid.
Fig. 38, pl. 10; No. 137, S. & J.
Brown powder. In solution brown, yellow.
7.78 g. per liter.
Angle 23.4′. Depth 0 to 0.21 mm.
Strong absorption in the violet and blue. A more concentrated solution exhibited absorption in the green and yellow. A long band or region of absorption extends from 0.35μ to 0.52μ. The maximum is near 0.395μ. There is a slight minimum of absorption at 0.48μ. The presence of a weaker, less refrangible band, increasing in intensity at 0.48μ, is more marked as the concentration is increased. More concentrated solutions show that the transparency in the ultra-violet rapidly disappears, whereas the bands do not encroach as rapidly on the yellow. Transparent from 0.53μ to 0.63μ. The absorption of the concentrated solutions is like that of solution No. 77, fig. 35.

41. Resorcine Brown—*Continued.*

42. Acid Brown. (D.) Sodium salt of bisulphobenzene-disazo-α-naphthol.
Similar to fig. 39, pl. 10; No. 138, S. & J.
Brown powder. In solution brown, yellow.
7.5 g. per liter.
Angle 25.5′. Depth 0 to 0.21 mm.
Absorption in violet and blue. Similar absorption to that of solution No. 8. Very weak absorption from 0.20μ to 0.29μ. Transparent to continuous background of spark from 0.29μ to 0.33μ. Weak, indefinite absorption band from 0.33μ to 0.48μ, with maximum indeterminate. Transparent to yellow, orange, and red.

43. Ponceau B O, extra. (A.) Sodium salt of benzene-azo-benzene-azo-β-naphthol-disulphonic acid.
Similar to fig. 52, pl. 13; No. 146, S. & J.
Light-brown powder. In solution yellowish red, pink.
7 g. per liter.
Angle 21.5′. Depth 0 to 0.20 mm.
Strong absorption in blue and green. Similar absorption to that of solution No. 21. Absorption decreases from 0.20μ to 0.295μ, and then increases to a maximum near 0.345μ. This band fades to semi-transparency about 0.4μ. The width and general appearance of the region of separation between the ultra-violet bands and the band in the green resembles much more closely the corresponding region for solution No. 48 than for solution No. 21. Transparency continues from 0.4μ to 0.44μ, where a strong, round band begins. The last band ends at 0.565μ. Its maximum is at 0.51μ. Transparent from 0.565μ to 0.63μ. Absorption from this band moves more rapidly towards the ultra-violet than towards the red, with increasing concentration. Same empirical formula as No. 42.

44. Janus Red B. (M.) Chloride of trimethyl-amido-benzene-azo-*m*-toluene-azo-β-naphthol.

Similar to fig. 23, pl. 6; No. 149, S. & J.

Reddish-brown powder. In solution yellowish red, faint yellowish red.

Saturated.

Angle 54.6′. Depth 0 to 0.50 mm.

Pointed, V-shaped, weak band in the blue-green. Similar absorption to that of solution No. 60. Absorption is strong at 0.20μ and decreases gradually and in a poorly defined manner to transparency at 0.375μ. The transparent region continues to 0.455μ. An absorption band lies between 0.455μ and 0.540μ, with its maximum at 0.51μ. Slight general absorption in the yellow and orange, but transparent to the red.

45. Cloth Red G. (O.) Sodium salt of toluene-azo-toluene-azo-β-naphthol-monosulphonic acid.

Fig. 25, pl. 7; No. 153, S. & J.

Reddish-brown powder. In solution red, faint pink.

8.33 g. per liter (heated and filtered).

Angle 31.2′. Depth 0 to 0.29 mm.

Bands in blue-green and green with hazy edges and weak general absorption in both directions. A band extends from 0.445μ to 0.55μ and appears to be composed of a stronger band with a weaker, more refrangible, unresolved companion. Their maximum of absorption is at 0.515μ. The slant at the end of the negative shows that weak, general absorption is exerted in the orange. Transparent to the red.

46. Cloth Red O. (M.) Sodium salt of toluene-azo-toluene-azo-β-naphthol-disulphonic acid.

Fig. 21, pl. 6; No. 154, S. & J.

Dark-brown powder. In solution deep red, very faint red.

6.36 g. per liter (warmed and filtered).

Angle 33.2′. Depth 0 to 0.30 mm.

Maximum of absorption in blue-green with weak absorption in yellow and orange. Absorption band starts at 0.485μ, attains its maximum at 0.52μ, and is dissipated in weak general absorption about 0.555μ. The end of the negative slants appreciably. Transparent to red.

47. Cloth Red 3 G A. (A.) Sodium salt of toluene-azo-toluene-azo-β-naphthylamine-monosulphonic acid.

Fig. 11, pl. 3; No. 155, S. & J.

Brownish-red powder. In solution reddish brown, light brown.

Saturated (heated).

Angle 1° 57′. Depth 0 to 1.07 mm.

General absorption in violet, also a maximum of absorption in the blue-green and green. Absorption is about complete at 0.20μ and increases very gradually, with hazy contour, to semi-transparency at about 0.42μ. Semi-transparency from 0.42μ to 0.49μ. Weak band from 0.49μ to 0.54μ. Transparent from 0.54μ to 0.63μ.

48. Ponceau 4 R B. (A.) Sodium salt of sulphobenzene-azo-benzene-azo-β-naphthol-monosulphonic acid.

Fig. 51, pl. 13; No. 160, S. & J.

Reddish-brown powder. In solution yellowish red, pink.

10 g. per liter (heated).

Angle 19.5′. Depth 0 to 0.18 mm.

Strong absorption in blue-green and green. At 0.455μ the strong band begins and extends to 0.565μ, with its maximum about 0.51μ. Transparent from 0.565μ to 0.63μ.

49. Biebrich Scarlet. (K.) Sodium salt of sulphobenzene-azo-sulphobenzene-azo-β-naphthol.

Similar to fig. 51, pl. 13; No. 163, S. & J.

Reddish-brown powder. In solution red, pink.

6 g. per liter.

Angle 19.5′. Depth 0 to 0.18 mm.

Absorption band in blue-green and green. Absorption decreases from 0.20μ to about 0.32μ, where a semi-transparent region appears. This is followed by an absorption band with its maximum at 0.355μ. Slight absorption from 0.39μ to 0.45μ. A definite band starts at 0.45μ, reaches a maximum near 0.51μ, and ends at 0.555μ. Transparent from 0.555μ to 0.63μ. A solution of 10 g. per liter showed almost complete absorption from 0.20μ to 0.36μ. From 0.395μ to 0.445μ only the first photographic strip received light. The band is very round from 0.445μ to its end at

49. Biebrich Scarlet—*Continued.*
0.585μ. Transparent to orange and red. Same empirical formula as No. 48 and almost identical visible absorption.

50. Wool Black. (A.) Sodium salt of sulphobenzene-azo-sulphobenzene-azo-*p*-tolyl-β-naphthylamine.
Fig. 67, pl. 17; No. 166, S. & J.
Bluish-black powder. In solution purple, light purple.
5.56 g. per liter (filtered).
Angle 35.1′. Depth 0 to 0.32 mm.
Hazy band in yellow spreading indefinitely into the orange. Transmits bright red. At 0.49μ a region of absorption commences which seems to consist of a hazy central band with a weak, washed-out companion on each side. The chief maximum is about 0.54μ. Absorption is very weak from 0.60μ to 0.63μ. A very concentrated solution shows that the maximum of transparency is near 0.44μ.

51. Ponceau 6 R B. (A.) Sodium salt of sulphotoluene-azo-toluene-azo-β-naphthol-α-sulphonic acid.
Fig. 56, pl. 14; No. 169, S. & J.
Reddish-brown powder. In solution scarlet red, pink.
5.38 g. per liter.
Angle 31.2′. Depth 0 to 0.29 mm.
Strong band in the green which is more definite on the blue side than on the yellow border. Strong band from 0.465μ to 0.565μ. The maximum is at 0.51μ. The band is probably composed of two unresolved bands of which the weaker lies nearer the red. Transparent from 0.565μ to 0.63μ.

52. Blue-Black. (B.) Sodium salt of sulpho-β-naphthalene-azo-α-naphthalene-azo-β-naphthol-disulphonic acid.
No. 186, S. & J.*
Bluish-black powder. In solution bluish violet, violet.
6.69 g. per liter (filtered).
Angle 33.2′. Depth 0 to 0.30 mm.
Very indefinite absorption in green-yellow, yellow, and orange, with maximum in yellow. Absorption is strong at 0.20μ and decreases gradually to about 0.34μ. Approximately transparent from 0.4μ to 0.5μ. Absorption starts near 0.50μ, increases to a maximum about 0.54μ, and decreases to weak, general absorption from 0.58μ to 0.63μ.

53. Anthracene Yellow C. (C.) Sodium salt of thio-di-benzene-disazo-di-salicylic acid.
Similar to fig. 37, pl. 10; No. 190, S. & J.
Brownish-yellow powder. In solution muddy yellowish brown, greenish yellow.
6 g. per liter (filtered).
Angle 21.3′. Depth 0 to 0.18 mm.
Absorption in violet. Somewhat similar absorption to that of solution No. 81. However, the absorption of solution No. 53 is more intense than that of solution No. 81. Absorption decreases from 0.20μ to a semi-transparent strip at about 0.295μ. Beyond this strip a band with hazy contour extends as far as 0.41μ with its maximum at about 0.34μ. Transparent from 0.41μ to 0.63μ.

54. Bismarck Brown. (A.) Hydrochloride of benzene-disazo-phenylene-diamine.
Fig. 7, pl. 2; No. 197, S. & J.
Dark-brown powder. In solution brown, yellow.
30 g. per liter (filtered).
Angle 19.5′. Depth 0 to 0.18 mm.
Visible and photographic absorption identical with that of solution No. 12.

55. Vesuvine. (B.) Hydrochloride of toluene-disazo-*m*-tolylene-diamine.
Fig. 7, pl. 2; No. 201, S. & J.
Dark-brown powder. In solution reddish brown, yellow.
4.29 g. per liter.
Angle 31.2′. Depth 0 to 0.29 mm.
Visible and photographic absorption identical with that of solution No. 12.

56. Congo Orange G. (A.) Sodium salt of diphenyl-disazo-phenetol-β-naphthylamine-disulphonic acid.
Fig. 33, pl. 9; No. 217, S. & J.
Brownish-red powder. In solution reddish yellow, yellow.
5.38 g. per liter.
Angle 23.4′. Depth 0 to 0.21 mm.
Hazy absorption in blue, blue-green, and green with maximum in the green. Tolerably strong absorption decreases from 0.20μ to a weak minimum near 0.32μ and then increases to a maximum about 0.36μ. A semi-transparent region lies between

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*Spectrogram too indefinite for reproduction.

56. Congo Orange G—*Continued.* 0.405μ and 0.44μ. A weak, hazy band begins at 0.44μ and continues to 0.475μ, at which point it joins a more intense band. The latter has its maximum at 0.505μ and then decreases to transparency at 0.53μ. Transparent from 0.53μ to 0.63μ. More concentrated solutions emphasize all the maxima of absorption just outlined and the minimum at 0.42μ.

57. Chrysamine G. (A.) Sodium salt of diphenyl-disazo-bi-salicylic acid.
Similar to fig. 36, pl. 9; No. 220, S. & J.
Yellowish-brown powder. In solution brownish yellow, faint yellow.
7 g. per liter (heated and filtered).
Angle 35.1′. Depth 0.26 to 0.58 mm.
No visible absorption unless, perhaps, a faint weakening of the extreme violet. Somewhat similar absorption to that of the more dilute solution of No. 77. For the layer used the absorption is nearly complete from 0.20μ to 0.29μ. The continuous background is transmitted from 0.29μ to 0.30μ. The limit of the ultra-violet absorption is approximately a straight line joining the wave-lengths 0.365μ and 0.395μ at the opposite edges of the negative. Transparent from 0.395μ to 0.63μ. More dilute solutions and wedges of liquid tapering to infinitesimal thickness show that ultra-violet absorption is very weak.

58. Cresotine Yellow G. (M.) Sodium salt of diphenyl-disazo-bi-*o*-cresol-carboxylic acid.
Similar to fig. 36, pl. 9; No. 221, S. & J.
Yellowish-brown powder. In solution yellow, faint yellow.
Saturated.
Angle 39.0′. Depth 0 to 0.36 mm.
Absorption in violet and indigo. Absorption much like that of the more dilute solution of No. 77. The solution has a peculiar odor. A washed-out band begins at 0.31μ, passes through a maximum near 0.355μ, and then fades away at 0.44μ. A rather narrow region of semi-transparency, the center of which is near 0.30μ, separates this band from a weak, more refrangible ultra-violet band. Transparent from 0.44μ to 0.63μ.

59. Congo Red. (A.) Sodium salt of diphenyl-disazo-bi-naphthionic acid.
Similar to fig. 26, pl. 7; No. 240, S. & J.
Reddish-brown powder. In solution red, yellowish red.
5.9 g. per liter.
Angle 27.3′. Depth 0 to 0.25 mm.
Absorption in blue, blue-green, and green with maximum nearer the green end. Similar absorption to that of solution No. 69. Absorption decreases from 0.20μ to near 0.28μ and then increases to a maximum at about 0.325μ. These two partially resolved bands are followed by a region of approximate transparency extending from 0.385μ to 0.426μ with its maximum at 0.405μ. Transparency is terminated at 0.426μ by a pair of wide, hazy bands of which the more refrangible is the weaker. The chief maximum is at 0.505μ. Absorption ends at 0.545μ. Transparent from 0.545μ to 0.63μ. More concentrated solutions show that the ultra-violet and visible bands soon run together, whereas the absorption does not advance much towards the orange.

60. Congo Corinth G. (A.) Sodium salt of diphenyl-disazo-naphthionic-α-naphthol-sulphonic acid.
Fig. 23, pl. 6; No. 242, S. & J.
Greenish-black powder. In solution brownish red, red.
5.38 g. per liter (heated).
Angle 27.3′. Depth 0 to 0.25 mm.
Absorption in blue-green, green, and yellow. Very general absorption at the red border. Absorption band from 0.46μ to 0.555μ with its maximum near 0.515μ. The end of the negative slants considerably, showing that general absorption continues into the orange. Transparent to red. Same empirical formula as No. 61, but different constitution.

61. Congo Rubine. (A.) Sodium salt of diphenyl-disazo-naphthionic acid-β-naphthol-sulphonic acid.
Similar to fig. 19, pl. 5; No. 243, S. & J.
Greenish, crystalline powder. In solution bluish red, dull red.
Saturated (warmed).
Angle 39.0′. Depth 0.26 to 0.62 mm.

61. Congo Rubine—*Continued.*
Absorption in blue-green. Absorption, especially in the visible spectrum, similar to that of solution No. 106. A layer about 2 mm. deep absorbs all the visible spectrum except the orange and red. Complete absorption at 0.20μ decreases to a minimum near 0.275μ, then increases to a maximum at about 0.31μ, and finally vanishes in transparency at 0.345μ. Absorption band from 0.485μ to 0.55μ with its maximum at 0.52μ. Transparent from 0.55μ to 0.63μ. Same empirical formula as No. 60, but different constitution.

62. Anthracene Red. (I.) Sodium salt of nitrodiphenyl-disazo-salicylic-α-naphthol-sulphonic acid.
Similar to fig. 56, pl. 14; No. 262, S. & J.
Brownish-red powder. In solution deep red, pink.
7.5 g. per liter.
Angle 27.3′. Depth 0 to 0.25 mm.
Hazy-edged band in the blue-green and green. Similar absorption to that of solution No. 51. Absorption decreases from 0.20μ to semi-transparency at 0.29μ. About 0.31μ a well-rounded, hazy-edged band starts, passes through its maximum near 0.36μ, and ceases at 0.42μ. Partial transparency from 0.42μ to 0.465μ. A symmetrical absorption band prevents transmission from 0.465μ to 0.55μ. Its maximum is near 0.51μ. Transparent from 0.55μ to 0.63μ. Less concentrated solutions show that the ultra-violet absorption is comparatively weak.

63. Congo Orange R. (A.) Sodium salt of ditolyl-disazo-phenetol-β-naphthylamine-disulphonic acid.
Similar to fig. 11, pl. 3; No. 275, S. & J.
Yellowish-red powder. In solution brown, yellowish brown.
5 g. per liter (warmed and filtered).
Angle 39.0′. Depth 0 to 0.36 mm.
Indefinite, general absorption in the blue and blue-green. The liquid is not clear, but behaves somewhat like an emulsion. Similar absorption to that of solution No. 47. Absorption decreases from 0.20μ to about 0.29μ and then remains about constant as far as 0.365μ. From this point it decreases to very weak, general absorption at 0.43μ. A slight increase in absorption has its maximum at 0.515μ. It is partly, but not entirely, due to the weak spot of the Seed emulsion. Transparent from 0.54μ to 0.63μ. Deeper layers of greater concentration show the minimum of absorption to be the region around 0.455μ.

64. Benzopurpurine 6 B. (A.) Sodium salt of ditolyl-disazo-bi-α-naphthylamine-sulphonic acid.
Similar to fig. 26, pl. 7; No. 278, S. & J.
Red powder. In solution red, brownish red.
7.78 g. per liter (filtered).
Angle 29.3′. Depth 0 to 0.27 mm.
Hazy-edged band in blue and green. Similar absorption to that of solution No. 69. Strong absorption from 0.415μ to 0.55μ. The chief maximum of absorption is at 0.51μ. Probably two hazy, unresolved bands, with more refrangible, weaker component. Transparent from 0.55μ to 0.63μ. Weaker solutions show more rapid increase of transparency on the ultra-violet side of the visible band than on the red side.

65. Benzopurpurine B. (A.) Sodium salt of ditolyl-disazo-bi-β-naphthylamine-β-sulphonic acid.
Similar to fig. 26, pl. 7; No. 279, S. & J.
Brown powder. In solution reddish brown, brown.
8.75 g. per liter.
Angle 25.4′. Depth 0 to 0.23 mm.
Same visible and photographic absorption as No. 69. Identical visible absorption to that of solution No. 64. Solutions Nos. 65 and 69 seem to have only one region of absorption in the ultra-violet, whereas solution No. 64 has a slight minimum of absorption near 0.275μ. Nos. 64 and 65 have the same empirical formulæ, but different chemical constitution.

66. Diamine Red B. (A.) Deltapurpurine 5 B. (M.) Sodium salt of ditolyl-disazo-bi-β-naphthylamine-sulphonic acid.
Similar to fig. 26, pl. 7; No. 280, S. & J.
Reddish-brown powder. In solution yellowish red, pink.

66. Diamine Red B—*Continued.*

6.36 g. per liter (warmed and filtered).

Angle 25.4′. Depth 0 to 0.23 mm.

Same visible absorption as No. 64. Similar absorption to that of solution No. 69. Nos. 64 and 66 have the same empirical formulæ, but different chemical constitutions.

67. Brilliant Congo R. (A). Sodium salt of ditolyl-disazo - β - naphthylamine-monosulphonic - β - naphthylamine-disulphonic acid.

Similar to fig. 26, pl. 7; No. 281, S. & J.

Brown powder. In solution, yellowish red, yellowish red.

8 g. per liter.

Angle 31.2′. Depth 0 to 0.29 mm.

Hazy-edged band in blue and green. Similar absorption to that of solution No. 69. A pair of unresolved bands extends from 0.42μ to 0.56μ, with their chief maximum at 0.515μ. The less refrangible band is the more intense. Transparent from 0.555μ to 0.63μ.

68. Diamine Red 3 B. (A.) Sodium salt of ditolyl-disazo-bi-β-naphthylamine-δ-sulphonic acid.

Fig. 61, pl. 16; No. 282, S. & J.

Reddish-brown powder. In solution reddish brown, brown.

7 g. per liter (heated and filtered).

Angle 50.7′. Depth 0.64 mm. to 1.10 mm.

Hazy-edged absorption in the blue-green and green A pair of unresolved bands absorbs from about 0.435μ to 0.545μ. Their maximum is near 0.515μ. The more refrangible band is weaker and more indefinite than its companion. Transparent from 0.545μ to 0.63μ.

69. Brilliant Purpurine R. (A.) Sodium salt of ditolyl-disazo-naphthionic-β-naphthylamine-disulphonic acid.

Fig. 26, pl. 7; No. 283, S. & J.

Red powder. In solution red, pink.

8.75 g. per liter.

Angle 25.4′. Depth 0 to 0.23 mm.

Same visible and photographic absorption as No. 67. Nos. 67 and 69 have the same empirical formulæ, but different chemical constitutions. Absorption gradually decreases from 0.20μ to partial transparency at 0.388μ. This minimum of absorption extends to 0.411μ. A pair of unresolved bands absorbs all radiations between 0.411μ and 0.563μ. The less refrangible band is the more intense and has its maximum at 0.515μ. Transparent from 0.563μ to 0.63μ.

70. Rosazurine B. (B.) Sodium salt of ditolyl - disazo - bi - ethyl-β-naphthylamine-sulphonic acid.

Similar to fig. 26, pl. 7; No. 285, S. & J.

Brown powder. In solution red, pink.

7 g. per liter (heated).

Angle 27.3′. Depth 0 to 0.25 mm.

Weak absorption in the green, with very hazy, blue boundary. Similar absorption to that of solution No. 69. The precise size and shape of the visible band matches very closely the corresponding band of solution No. 45, fig. 25. Absorption from 0.47μ to 0.55μ, with the maximum near 0.515μ. Transparent from 0.55μ to 0.63μ.

71. Congo Corinth. Sodium salt of ditolyl-disazo - naphthionic-α-naphthol-*p*-sulphonic acid.

Similar to fig. 22, pl. 6; No. 286, S. & J.

Grayish-black powder. In solution red, pink.

9.09 g. per liter (very gritty; filtered often).

Angle 21.5′. Depth 0 to 0.20 mm.

General absorption in green, yellow, and orange. Very weak and hazy towards the red. Similar absorption to that of solution No. 74. Weak, general absorption in ultra-violet, permitting all strong lines to pass through the solution. It fades away about 0.35μ. Several plates and films show that absorption begins again about 0.49μ and becomes relatively small at 0.56μ. Weak, general absorption, however, continues to 0.63μ. The maximum of absorption is indeterminate. Saturated solutions gave the same general results.

72. Azo Blue. (By.) Sodium salt of ditolyl-disazo-bi-α-naphthol-*p*-sulphonic acid.

Somewhat like fig. 22, pl. 6; No. 287, S. & J.

Bluish-black powder. In solution deep blue, reddish blue.

4.29 g. per liter.

Angle 23.4′. Depth 0 to 0.21 mm.

Hazy-edged absorption in green-yellow, yellow, and orange. The red border

72. Azo Blue—*Continued.*
is very indefinite and the maximum appears to be in the yellow. Absorption is like that of solution No. 74, except in so far as it is stronger than No. 74 in the orange. Comparatively weak absorption in the ultraviolet from 0.20μ to 0.355μ. There are slight indications of two ultraviolet bands with the intervening region near 0.29μ. Transparent from 0.36μ to 0.49μ. Absorption begins at 0.49μ, increases to 0.53μ, then fades to partial transparency near 0.59μ. Weak absorption from 0.59μ to 0.63μ.

73. Diamine Black B O. (C.) Sodium salt of ethoxy-diphenyl-disazo-bi-amidonaphthol-sulphonic acid.
Suggested by fig. 22, pl. 6; No. 304, S. & J.
Black powder. In solution blue, blue.
7.5 g. per liter.
Angle 31.2′. Depth 0 to 0.29 mm.
Strong absorption in the yellow and orange with diffuse borders. Red is transmitted. The absorption is somewhat like that of solution No. 74 except in so far as it is stronger than No. 74 in the orange. Absorption decreases from 0.20μ to about 0.37μ. Transparent from 0.37μ to 0.51μ. Absorption extends from 0.51μ into the clear red. The maximum of absorption is indefinite.

74. Benzopurpurine 10 B. (A.) Sodium salt of dimethoxy-di-phenyl-disazo-bi-naphthionic acid.
Fig. 22, pl. 6; No. 307, S. & J.
Brownish-red powder. In solution red, pink.
5.83 g. per liter.
Angle 23.4′. Depth 0.05 to 0.26 mm.
Chief absorption in the green. The visible band extends from about 0.48μ to 0.55μ with its maximum at 0.515μ. The slanting end of the negative denotes general absorption in the orange.

75. Benzoazurine. Sodium salt of dimethoxy-diphenyl disazo-bi-α-naphthol-*p*-sulphonic acid.
Somewhat like fig. 22, pl. 6; No. 311, S. & J.
Bluish-black powder. In solution reddish blue, blue.
7.5 g. per liter.

75. Benzoazurine—*Continued.*
Angle 25.5′. Depth 0 to 0.21 mm.
No very definite, visible band, but a general weakening of the green, yellow, orange, and red. The ultraviolet absorption is weak and ends near 0.355μ. The region of general absorption begins about 0.51μ and continues beyond 0.63μ. There is a weak maximum near 0.53μ. The end of the negative slants to an unusual degree. The contour of the weak band is V-shaped like the visible band for solution No. 74.

76. Diamine Green B. (C.) Sodium salt of diphenyl-disazo-phenol-disulpho-amidonaphthol-azo-nitrobenzene.
No. 372, S. & J.*
Dull-gray, crystalline powder. In solution bluish green, bluish green.
3 g. per liter.
Angle 17.0′. Depth 0 to 0.14 mm.
Gradual absorption in the orange and red. Very weak absorption in the ultra-violet from 0.20μ to about 0.38μ. No visible or photographic absorption between 0.38μ and 0.6μ. General absorption begins about 0.6μ.

77. Congo Brown G. (A.) Sodium salt of sulpho-benzene-azo-resorcinol-azo-diphenyl-azo-salicylic acid.
Figs. 35 and 36, pl. 9; No. 379, S. & J.
Brown powder. In solution light brown, yellow.
4.67 g. per liter.
Angle 27.3′. Depth 0 to 0.25 mm.
Absorption in violet, blue, and green. Very hazy at green side. Absorption decreases from 0.20μ to a somewhat transparent region around 0.295μ. Maximum absorption at 0.365μ. Absorption ceases at 0.53μ. A weaker solution (fig. 36) showed transparency from 0.29μ to 0.315μ. Absorption from 0.315μ to 0.428μ. Maximum absorption at 0.365μ, as before. Transparent to yellow, orange, and red.

78. Congo Brown R. (A.) Sodium salt of sulpho-naphthalene-azo-resorcin-azo-diphenyl-azo-salicylic acid.
Fig. 35, pl. 9; No. 380, S. & J.
Dark, brownish-red powder. In solution reddish brown, yellow.
Saturated.
Angle 29.3′. Depth 0 to 0.27 mm.

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*Spectrogram too indefinite for reproduction.

78. Congo Brown R—*Continued.*
Same visible and photographic absorption as No. 77. The change in the constitution is from benzene to naphthalene.

79. Fast Green O. (M.) Dinitroso-resorcinol. (Dioximidoquinone.)
Similar to fig. 11, pl. 3; No. 394, S. & J.
Grayish-brown powder. In solution deep coffee brown, coffee brown.
Saturated (heated).
Angle 1° 6′. Depth 0.05 to 0.66 mm.
General absorption in violet and blue. Similar absorption to that of solution No. 47. The boundaries of the bands, however, are more definite for solution No. 79 than for solution No. 47. From 0.20μ to 0.325μ the absorption is complete. Absorption decreases with a long, gradual slope, from 0.325μ to a minimum of semi-transparency at 0.475μ. Then a weak band with maximum at 0.52μ presents itself and continues to 0.54μ. Only very weak absorption is present from 0.54μ to 0.63μ. With the same solution and the cell set for 35.1′ and 0.32 mm. the absorption was almost complete from 0.20μ to 0.30μ and then sloped gradually to transparency at 0.405μ. The band at 0.52μ could not be discerned.

80. Naphthol Green B. (C.) Ferrous sodium salt of nitroso-β-naphthol-β-monosulphonic acid.
Fig 10, pl. 3; No. 398, S. & J.
Dark-green powder. In solution green, light green.
8.75 g. per liter (boiled).
Angle 31.2′. Depth 0 to 0.29 mm.
Absorption in violet and in dark red. Absorption very strong from 0.20μ to 0.31μ. Then the absorption decreases very gradually, with a long slant, to 0.455μ. From 0.455μ to the orange the transparency is complete. Absorption begins again in the red.

81. Curcumine S. (A.) Sodium salt of the so-called azoxy-stilbene-disulphonic acid.
Fig. 37, pl. 10; No. 399, S. & J.
Brown powder. In solution yellow, faint yellow.
Saturated.
Angle 1° 10′. Depth 0.11 to 0.75 mm.

81. Curcumine S—*Continued.*
Faint absorption in violet. Absorption decreases from 0.20μ to 0.29μ. Transparent from 0.29μ to 0.34μ. Weak band from 0.34μ to 0.43μ with maximum at 0.39μ. Transparent from 0.43μ to 0.63μ.

82. Auramine O. (B.) Hydrochloride of amido-tetramethyl-diamido-diphenyl-methane.
Fig. 43, pl. 11; No. 425, S. & J.
Sulphur-yellow powder. In solution yellow, faint yellow.
Equal volumes of a saturated solution and of water (filtered).
Angle 42.5′. Depth 0 to 0.36 mm.
Strong absorption in violet and indigo. Relatively transparent at 0.22μ. The continuous background of the spark indicates one band or, at most, two bands from 0.23μ to 0.275μ. Unusually transparent from 0.275μ to 0.345μ. A pair of partially-resolved intense bands absorb from 0.345μ to 0.47μ. Their maxima lie at 0.365μ and 0.425μ. The intervening, partially-transparent spot is near 0.385μ. The less refrangible band is the more intense and is very round. Transparent from 0.470μ to 0.63μ.

83. Malachite Green. (M.) Oxalate of tetramethyl-di-*p*-amido-triphenyl-carbinol.
Similar to fig. 46, pl. 12; No. 427, S. & J.
Green, metallic, glistening plates. In solution greenish blue, greenish blue.
3.75 g. per liter.
Angle 25.5′. Depth 0 to 0.21 mm.
Strong, double band in the orange and clear red. Deep red is transmitted. Similar absorption to that of solution No. 86 from 0.20μ to the yellow. All strong lines in the extreme ultra-violet are transmitted freely. Absorption band lies between 0.29μ and 0.33μ. A faint band has its maximum near 0.425μ. Strong absorption commences at 0.55μ.

84. Emerald Green. (B.) Sulphate or zinc-double-chloride of tetraethyl-diamido-triphenyl-carbinol.
Similar to fig. 46, pl. 12; No. 428, S. & J.
Golden, glistening crystals. In solution green, green.
4.62 g. per liter.

84. Emerald Green—*Continued.*
Angle 31.2′. Depth 0 to 0.29 mm.
Compound band in the orange and red with the maximum in the red. The contour is hazy. Similar absorption to solution No. 86 from 0.20μ to the yellow, save that the band near 0.425μ is hardly discernible on the negative. Strong absorption begins at 0.57μ. For ultra-violet details see No. 83 above.

85. Light Green F S. (B.) Sodium salt of dimethyldibenzyl-diamido-triphenyl-carbinol-trisulphonic acid.
Similar to fig. 46, pl. 12; No. 434, S. & J.
Brownish-black powder. In solution green, green.
10 g. per liter.
Angle 21.3′. Depth 0 to 0.18 mm.
Strong band in the yellow, orange, and red. Except for concentration, the absorption is the same as that of solution No. 86, hence for further details refer to No. 86.

86. Acid Green, concentrated. (C.) Sodium salt of diethyldibenzyl-diamido-triphenyl-carbinol-trisulphonic acid.
Fig. 46, pl. 12; No. 435, S. & J.
Bright-green, dull powder. In solution deep green, green.
13.33 g. per liter.
Angle 25.4′. Depth 0 to 0.23 mm.
Strong band in orange and red with no return to transparency visible. Absorption in violet and blue. Absorption decreases from 0.20μ to about 0.275μ. Then a strong band begins, having its maximum near 0.32μ and returning abruptly to transparency at 0.34μ. Transparent from 0.34μ to 0.39μ. A round band extends from 0.39μ to 0.455μ with its maximum at 0.425μ. Transparent from 0.455μ to 0.55μ. Strong absorption commences at 0.55μ and increases to complete opacity at 0.63μ. Weaker solutions show conclusively the transparent region around 0.275μ and also that the band at 0.425μ vanishes most readily.

87. Fuchsine. (M.) Mixture of hydrochloride or acetate of pararosaniline and rosaniline.
Fig. 48, pl. 12; No. 448, S. & J.
Green, crystalline powder. In solution deep red, red.

87. Fuchsine—*Continued.*
Angle 21.3′. Depth 0 to 0.18 mm.
Intense band in blue-green and green. All lines near 0.23μ and from 0.25μ to 0.26μ are freely transmitted. The background indicates a band with its maximum at 0.285μ and extending from 0.27μ to 0.305μ. Transparent from 0.305μ to 0.45μ. Very strong absorption from 0.45μ to 0.575μ with maximum near 0.53μ. There are probably two unresolved bands of which the more refrangible is the weaker. Transparent from 0.575μ to 0.63μ. A layer about 1 mm. deep limited the more refrangible, transparent region to the interval from 0.35μ to 0.39μ.

88. New Magenta. (O.) Hydrochloride of triamido-tritolyl-carbinol.
Fig. 50, pl. 13; No. 449, S. & J.
Beetle-green powder. In solution red, bluish red.
6 g. per liter.
Angle 31.2′. Depth 0 to 0.29 mm.
Strong band in the green, steeper on the yellow side, and suggesting a sharp band superposed upon a weaker one. The band extends from about 0.44μ to 0.56μ with its maximum near 0.52μ. Transparent from 0.56μ to 0.63μ.

89. Dahlia. (B.) Mixture of the hydrochlorides or acetates of the mono- di- or tri-methyl (or ethyl) rosanilines and pararosanilines.
Fig. 69, pl. 18; No. 450, S. & J.
Green, lumpy powder. In solution deep blue, reddish violet.
3.57 g. per liter.
Angle 23.4′. Depth 0 to 0.21 mm.
Absorption commences in the blue-green, has its maximum in the green-yellow, and decreases gradually into the red. Transparent to deep red. Absorption in ultra-violet is weak. A band which is definite on the more refrangible edge commences at 0.48μ and increases to a maximum at 0.52μ. A weak, unresolved companion joins the last one near 0.57μ and fades away at 0.62μ.

90. Crystal Violet. (B.) Hydrochloride of hexamethyl-pararosaniline.
Similar to fig. 66, pl. 17; No. 452, S. & J.

90. Crystal Violet—*Continued.*

Cantharides glistening crystals. In solution violet, violet.

1.38 g. per liter.

Angle 42.5′. Depth 0 to 0.36 mm.

The absorption is the same as that exhibited by solution No. 92, hence see the description given below.

91. Ethyl Violet. (B.) Hydrochloride of hexaethyl pararosaniline.

Fig. 64, pl. 16; No. 453, S. & J.

Green, crystalline powder. In solution pure, deep blue, pinkish blue.

2.73 g. per liter.

Angle 27.3′. Depth 0 to 0.25 mm.

Intense absorption in the green. Very sharp and abrupt on the blue side but indefinite on the red border. Strong lines in the ultra-violet are transmitted pretty freely. Relative transparency in the vicinity of 0.265μ. Absorption ceases about 0.31μ and from this wave-length to 0.495μ transparency obtains. At 0.495μ an intense band having its maximum at 0.525μ begins. About 0.585μ the absorption becomes relatively weak and diffuse and continues thus to 0.63μ.

92. Methyl Violet 6 B. (A.) Chiefly a mixture of the hydrochlorides of benzylpentamethylpararosaniline and hexamethylpararosaniline.

Fig. 66, pl. 17; No. 454, S. & J.

Metallic, glistening powder. In solution blue, reddish blue.

2.5 g. per liter.

Angle 31.2′. Depth 0 to 0.29 mm.

Transmits only blue and pure red in concentrated solution. The solution described below showed two bands, the more intense in the green-yellow and the weaker in the orange. All strong ultra-violet lines are transmitted. The chief band starts at 0.50μ, has its maximum at 0.535μ, and joins its companion about 0.57μ. The weaker band has its maximum at 0.595μ and fades away at 0.615μ. Transparent from 0.62μ to beyond 0.63μ.

93. Methyl Green OO. (By.) Zinc-double-chloride of hepta-methyl-pararosaniline-chloride.

Similar to fig. 47, pl. 12; No. 460, S. & J.

Green crystals. In solution greenish blue, greenish blue.

93. Methyl Green OO—*Continued.*

6 g. per liter.

Angle 19.5′. Depth 0 to 0.18 mm.

Band in violet and blue, also strong absorption in orange and red. The band in the orange is partly separated from the stronger band whose intensity increases to, and beyond, 0.63μ. With due allowance for differences in concentration, it appears that the absorption is, in toto, the same as that shown by solution No. 94. Absorption decreases from 0.20μ to a region of less intense absorption at 0.28μ. Then follows a strong band with maximum at 0.315μ. A band from 0.375μ to 0.445μ has its maximum at 0.415μ. The yellow and orange band begins at 0.515μ.

94. Methyl Green. Zinc-double-chloride of ethylhexamethyl-pararosaniline bromide.

Fig 47, pl. 12; No. 461, S. & J.

Moss-green, crystalline powder. In solution bluish green, bluish green.

20 g. per liter.

Angle 19.5′. Depth 0 to 0.18 mm.

Band in violet and blue, also strong absorption in orange and red. Transmits green and yellow-green. Band from 0.36μ to 0.45μ with maximum at 0.415μ. This band is steeper on its green side. Very transparent from 0.45μ to 0.495μ. From 0.495μ the absorption is intense and no return to transparency can be seen at 0.63μ. The band at 0.415μ disappears first on dilution.

95. Fuchsine S. (B.) Mixture of the sodium or ammonium salts of the trisulphonic acids of rosaniline and pararosaniline.

Fig. 53, pl. 14; No. 462, S. & J.

Metallic, green, glistening powder. In solution red, red.

3 g. per liter.

Angle 23.4′. Depth 0 to 0.21 mm.

Strong absorption in the blue-green and green. The middle of the first transparent region is about 0.265μ. A weak absorption band has its maximum near 0.295μ. The visible band commences at 0.475μ, has its maximum about 0.535μ, and ends at 0.575μ. Transparent from 0.575μ to 0.63μ.

96. Red Violet 5 R S. (B.) Sodium salt of ethylrosaniline-sulphonic acid.
Fig. 62, pl. 16; No. 463, S. & J.
Brownish-violet, metallic, glistening lumps. In solution brownish red, brownish red.
Saturated.
Angle 58.5′. Depth 0 to 0.54 mm.
General absorption in the green, yellow, and orange. The ultra-violet absorption is complete from 0.20μ to 0.36μ. Absorption decreases gradually from 0.36μ to a minimum of general absorption near 0.47μ. A weak band has its maximum about 0.525μ. The marked slant of the end of the negative shows the presence of appreciable general absorption in the yellow and orange. Red is transmitted.

97. Alkali Blue 6 B. (A. A. C.) Sodium salt of triphenyl-*p*-rosaniline-monosulphonic acid.
Fig. 72, pl. 18; No. 476, S. & J.
Blue powder. In solution blue, delicate blue.
12.5 g. per liter (heated).
Angle 23.4′. Depth 0 to 0.21 mm.
Strong absorption in yellow, orange, and red. Intense, continuous absorption from 0.20μ to 0.32μ. Abrupt decrease in absorption from 0.32μ to transparency at 0.345μ. Transparent from 0.345μ to 0.51μ. Strong absorption from 0.51μ to beyond 0.63μ. No decrease in absorption as far as 0.63μ. The apparent increase in absorption near 0.62μ is due to the relative diminution of sensitiveness of the photographic emulsion.

98. Methyl Blue. (O.) Sodium salt of triphenyl-pararosaniline-trisulphonic acid.
Similar to fig. 71, pl. 18; No. 479, S. & J.
Dark-blue powder. In solution deep, bright blue, blue.
6.89 g. per liter.
Angle 21.3′. Depth 0 to 0.18 mm.
Strong absorption in the yellow, orange, and red. The description for solution No. 99 holds here quantitatively.

99. China Blue. (A.) Sodium salt of the trisulphonic acid of triphenylrosaniline and triphenylpararosaniline.
Fig. 71, pl. 18; No. 480, S. & J.

99. China Blue—*Continued.*
Coppery flakes. In solution blue, blue.
3.57 g. per liter (filtered).
Angle 23.4′. Depth 0 to 0.21 mm.
A hazy-edged band begins in the green and continues into the red. The strong lines around 0.255μ are transmitted by the deepest layer of liquid. Absorption is more or less uniform from 0.20μ to 0.32μ and then shades off to transparency at 0.345μ. Transparent from 0.345μ to 0.505μ. Then a band starts and continues with undiminished intensity to 0.63μ.

100. Coralline Red. Dioxy-amido-triphenyl-carbidrid.
Fig. 49, pl. 13; No. 484, S. & J.
Reddish-brown lumps. In solution red, salmon pink.
11.25 g. per liter (heated).
Angle 19.5′. Depth 0 to 0.18 mm.
Sharp band in blue and green. Very definite at yellow side with maximum in green-yellow. Rather strong absorption from 0.20μ to 0.27μ, then a decrease to transparency at 0.315μ. Strong absorption from 0.445μ to 0.557μ with maximum at 0.527μ. Probably two unresolved bands with the weaker component at the more refrangible side. Transparent from 0.557μ to 0.63μ.

101. Night Blue. (B.) Hydrochloride of *p*-tolyltetraethyl-triamido-diphenyl-α-naphthyl-carbinol.
Similar to fig. 70, pl. 18; No. 489, S. & J.
Violet, bronzy powder. In solution bright blue, blue.
2.31 g. per liter.
Angle 23.4′. Depth 0 to 0.21 mm.
Absorption in the yellow, orange, and red. The limit at the green side is steep and the curve is flat in the longer wave-lengths. The visible region of absorption begins about 0.525μ. For the ultra-violet absorption see fig. 70.

102. Victoria Blue 4 R. (B.) Hydrochloride of phenylpenta-methyl-triamido-diphenyl-α-naphthyl-carbinol.
Fig. 70, pl. 18; No. 490, S. & J.
Bronzy, glistening powder. In solution deep blue, reddish blue.
2.78 g. per liter (heated).
Angle 27.3′. Depth 0 to 0.25 mm.

102. Victoria Blue 4 R—*Continued.*

Hazy-edged absorption in yellow and orange. The red is only partially absorbed. Comparatively weak absorption decreases steadily from 0.20μ to about 0.335μ. A band commences at 0.495μ and continues to a maximum at 0.53μ. From this point the band shades off gently towards the red. (The apparent increase of absorption at 0.62μ is probably due to the increasing lack of sensitiveness of the plate.)

103. Rhodamine B. (B.) Hydrochloride of diethyl-*m*-amido-phenol-phthaleïne.

Fig. 65, pl. 17; No. 504, S. & J.

Reddish-violet powder. In solution bluish red, violet.

7.5 g. per liter.

Angle 42.5′. Depth 0 to 0.36 mm.

Two distinct bands, the one in the yellow-orange and the other in the green-yellow. Eye observations, changes in concentration, and different makes of films show that the more refrangible band is the more intense. Fluorescent solution. Absorption decreases gradually from 0.20μ to an indefinite limit near 0.32μ. Strong absorption from 0.494μ to about 0.59μ. The maxima are at 0.524μ and 0.557μ with the intervening minimum of absorption at 0.54μ. Transparent beyond the orange band into the red.

104. Fast Acid Violet B. (M.) Sodium salt of diphenyl-*m*-amido-phenol-phthaleïn sulphonic acid.

Fig. 63, pl. 16; No. 506, S. & J.

Maroon powder. In solution bluish red, pink.

3.33 g. per liter.

Angle 27.3′. Depth 0 to 0.25 mm.

Absorption band in the green-yellow. This band is comparatively definite on the green side and has a shadowy companion on the red side. General absorption continues well into the orange-red. All strong lines in the ultra-violet are transmitted. The ultra-violet absorption ends about 0.33μ. The visible band begins at 0.505μ and has its maximum near 0.53μ. The less refrangible limit is indeterminate. The essential difference between the spectrograms for solutions Nos. 104 and 106 is that for the former the visible band is asymmetric, whereas for the latter it is symmetric.

105. Fast Acid Violet A 2 R. (M.) Sodium salt of di-*o*-tolyl-*m*-amido-phenol-phthaleïn-sulphonic acid.

Similar to fig. 19, pl. 5; No. 507, S. & J.

Violet-red powder. In solution red, pink.

4.67 g. per liter.

Angle 23.4′. Depth 0 to 0.21 mm.

Very narrow, definite band in the green. All strong ultra-violet lines are transmitted. The ultra-violet absorption ends about 0.335μ. The visible band begins at 0.495μ, has its maximum at 0.525μ, and ends near 0.57μ. The slanting end of the spectrogram indicates general absorption in the deep yellow. The green band for this solution is of the same type as the corresponding one for No. 104, although it is much less asymmetric, and therefore it resembles more closely No. 106. The spectrum of solution No. 105 is best described as a transition form between Nos. 104 and 106. Solutions Nos. 104 and 105 have the same empirical formulæ.

106. Acid Rosamine A. (A.) Sodium salt of di-mesidyl-*m*-amido-phenol-phthaleïn-sulphonic acid.

Fig. 19, pl. 5; No. 508, S. & J.

Light-red powder. In solution red, pink.

10 g. per liter.

Angle 29.3′. Depth 0 to 0.27 mm.

Single V-shaped band in the green. Weak absorption beginning with extinction at 0.20μ and fading gradually to transparency at 0.32μ. Absorption band covers the interval from 0.505μ to 0.565μ with its maximum at 0.535μ. Transparent from 0.565μ to 0.63μ.

107. Uranine. (B.) Sodium or potassium salt of fluoresceïne.

Figs. 15 and 16, pl. 4; No. 510, S. & J.

Yellowish-brown powder. In solution reddish yellow, yellow.

Intense, narrow band in the blue-green with a weaker companion on its more refrangible side. Very strong, yellowish-green fluorescence.

107. Uranine—*Continued.*

Fig. 16 resulted from a solution of 2 g. per liter.

Angle 23.4′. Depth 0 to 0.21 mm. Only the stronger band shows. Here it extends from 0.480μ to 0.504μ with its maximum at 0.493μ.

Fig. 15 corresponds to 2.67 g. per liter. The angle 31.2′ gives a maximum depth of 0.49 mm. Complete transparency from 0.330μ to 0.443μ. The spectrogram shows that the visible region of absorption has roughly parallel sides which are very definite. The visible maximum is at 0.493μ as before. At the outer edge of the fifth strip the absorption covers the interval from 0.443μ to 0.515μ.

A solution of 5 g. per liter, of angle 42.5′, and of depth 0 to 0.36 mm., absorbed from 0.432μ to 0.518μ with the maximum at 0.493μ.

A solution of 20 g. per liter with an angle of 42.5′ caused the ultra-violet and visible absorption bands to coalesce, on the third photographic strip, in a semi-transparent region extending roughly from 0.355μ to 0.395μ. Intense absorption from 0.395μ to 0.533μ. The short wave-length boundary is indefinite, but the opposite limit is very sharply defined and steep. Maximum absorption at 0.493μ.

Tests were made to ascertain whether or not the conditions were favorable to contamination of the absorption spectra by the fluorescent light. The most dilute solution was illuminated with intense ultra-violet light and an exposure of five minutes was given to the photographic film. Full development of the film brought out no trace of previously incident light. Therefore, since the Nernst glower alone was used in making the records of the visible bands and because, in all cases except one, more concentrated solutions were used, it follows that the spectrograms are correct representations of the absorption, at least so far as the fluorescent light is concerned.

108. Eosine, yellowish. (A.) Alkali salts of tetrabromo-fluoresceïne.

Fig. 58, pl. 15; No. 512, S. & J.

108. Eosine, yellowish—*Continued.*

Deep red powder. In solution yellowish red, pink.

20 g. per liter.

Angle 21.3′. Depth 0 to 0.18 mm.

Very strong absorption in blue and green. Faint green fluorescence increasing with dilution. Intense absorption from 0.20μ to 0.33μ. The absorption then decreases, first gradually and then steeply, to partial transparency at 0.37μ. This transparent region continues as far as 0.434μ. Intense absorption from 0.434μ to 0.56μ. From 0.515μ to 0.525μ the solution is almost opaque. Two unresolved bands seem to be present. The more refrangible boundary of the visible absorption is less definite than the opposite side. The latter limit is steep and sharp. Transparent from 0.56μ to 0.63μ.

109. Eosine à l'alcool. (B.) Potassium salt of tetrabromo-fluoresceïne-ethyl-ether.

Fig. 17, pl. 5; No. 514, S. & J.

Brown powder mixed with small, green crystals. In solution red, pink.

4.29 g. per liter (heated).

Angle 46.8′. Depth 0 to 0.43 mm.

Sharp, narrow band in green, abrupt on yellow side and diffuse on the blue side due to a faint companion band. Slight greenish-yellow fluorescence. Very weak absorption in extreme ultra-violet. Absorption begins at 0.495μ and ends at 0.540μ. The chief maximum is at 0.525μ. Transparent from 0.540μ to 0.63μ.

110. Methyl Eosine. (A.) Potassium salt of dibromodinitro-fluoresceïne.

Similar to fig. 58, pl. 15; No. 515, S. & J.

Brown, crystalline powder. In solution red, orange.

10 g. per liter.

Angle 15.6′. Depth 0 to 0.14 mm.

Intense band in blue and green. Apparently two unresolved bands. Similar absorption to that of solution No. 108. Absorption is rather complete from 0.20μ to 0.30μ and then decreases to about 0.36μ. Strong absorption from 0.46μ to 0.56μ. The principal maximum is at 0.52μ. Very transparent from 0.56μ to 0.63μ.

111. Eosine, bluish. (B.) Sodium salt of tetraiodo-fluoresceïne.
Similar to fig. 58, pl. 15; No. 517, S. & J.
Lavender powder. In solution red, pink.
15 g. per liter.
Angle 21.3′. Depth 0 to 0.18 mm.
Intense absorption in the blue-green. Similar absorption to that of solution No. 108. The yellowish-green fluorescence only appears in dilute solutions. Absorption decreases gradually from 0.20μ to 0.36μ. Intense absorption from 0.455μ to 0.555μ. Maximum of absorption at 0.52μ. There seem to be two unresolved bands of which the more refrangible is a little less intense than its companion. This region of absorption is sharper and steeper at its yellow border. Transparent from 0.555μ to 0.63μ.

112. Erythrosine. (M.) Sodium salt of tetraiodo-fluoresceïne.
Fig. 59, pl. 15; No. 517, S. & J.
Dark-red powder. In solution red, pink.
15 g. per liter.
Angle 21.3′. Depth 0 to 0.18 mm.
Intense absorption in the blue and green. A strong band in the middle with a slightly weaker, unresolved companion on each side. The yellowish-green fluorescence only appears in dilute solutions. The region of visible absorption is from 0.455μ to 0.562μ. The chief maximum is about 0.518μ. The yellow edge of this group of bands is very sharply defined. Transparent from 0.562μ to 0.63μ.

113. Cyanosine. (M.) Alkaline salt of tetra-bromodichloro-fluoresceïne-methyl-ether.
Fig. 18, pl. 5; No. 519, S. & J.
Brownish-red powder. In solution red, bluish pink.
Saturated (boiled).
Angle 25.4′. Depth 0 to 0.23 mm.
Band in blue-green and green. Faint, yellowish fluorescence. Strong absorption from 0.493μ to 0.553μ with its maximum at 0.525μ. Transparent from 0.553μ to 0.63μ.

114. Rose Bengal. (B.) Alkaline salt of tetraiododichloro-fluoresceïne.
Similar to fig. 59, pl. 15; No. 520, S. & J.
Brown powder. In solution red, orange.
15 g. per liter.
Angle 21.3′. Depth 0 to 0.18 mm.
Double, unresolved pair of bands in green. In amyl alcohol the bands were nearly resolved. Strong absorption from 0.20μ to about 0.33μ, except a slight weakening at 0.295μ. Transparent from 0.33μ to 0.48μ. Strong absorption from 0.48μ to 0.575μ. Transparent from 0.575μ to 0.63μ.

115. Phloxine. (B.) Sodium salt of tetrabromotetra-chloro-fluoresceïne.
Fig. 60, pl. 15; No. 521, S. & J.
Brick-red powder. In solution cherry red, pink.
12.5 g. per liter.
Angle 42.5′. Depth 0 to 0.36 mm.
Intense absorption in the green. Dark-green fluorescence. Visible band lies between 0.458μ and 0.57μ. The maximum is near 0.525μ. For dilute solutions the absorption is very much like that shown by fig. 18, except that the contour of the band in the green is sharper than for solution No. 113.

116. Galleïne. (By.) Pyrogallol-phthaleïn.
Similar to fig. 11, pl. 3; No. 525, S. & J.
Violet brown powder. In solution very dark brown, brown.
Saturated (heated).
Angle 58.5′. Depth 0 to 0.54 mm.
Hazy-edged absorption in the blue-green and green. Absorption is comparatively strong at 0.20μ and decreases very gradually to semi-transparency about 0.435μ. Only partial transparency exists between 0.435μ and 0.485μ. A V-shaped band absorbs from 0.485μ to 0.553μ. The maximum is about 0.52μ. The end of the negative slants a good deal, showing general absorption in the yellow-orange. The spectrogram is like that which would be obtained with a more concentrated solution of No. 47 if it were possible to produce such a condition.

117. Phosphine. (M.) Nitrate of chrysaniline (unsym. diamido-phenyl-acridine) and homologues.
Fig. 32, pl. 8; No. 532, S. & J.
Orange-yellow powder. In solution brown, yellow.
11.25 g. per liter (heated and filtered).
Angle 27.3′. Depth 0 to 0.25 mm.
Absorption in violet, blue, and green with hazy limits. Strong absorption from 0.20μ to 0.295μ, then weakening to semi-transparency at 0.325μ. Next a band with maximum at 0.36μ. Return to partial transparency at 0.41μ. Then follow two unresolved bands with maxima about 0.458μ and 0.50μ. Complete transparency from 0.52μ to 0.62μ. A solution so concentrated as to absorb all the ultra-violet and visible spectrum from 0.20μ to 0.538μ was transparent to 0.63μ.

118. Alizarine Brown. (M.) Trioxyanthraquinone.
Similar to fig. 11, pl. 3; No. 538, S. & J.
Dark-brown powder. In solution dull brown, brown.
Saturated.
Angle 35.1′. Depth 0 to 0.32 mm.
General, indefinite absorption except in the red. Absorption intense and uniform from 0.20μ to 0.33μ. From 0.33μ the absorption decreases very gradually and nearly linearly to about 0.47μ. A very weak band with its maximum at 0.52μ exists over and above the intensity minimum of the sensitized film. The end of the negative slopes appreciably, denoting continued general absorption in the orange. No visible weakening of the red. A maximum of transparency is around 0.48μ. The spectrograms for solutions Nos. 116 and 118 are very similar.

119. Alizarine Red S. (B.) Sodium salt of alizarine-monosulphonic acid.
Fig. 14, pl. 4; No. 546, S. & J.
Orange-yellow powder. In solution reddish yellow, yellow.
12 g. per liter (heated and filtered).
Angle 30.0′. Depth 0 to 0.45 mm.

119. Alizarine Red S—*Continued.*

Absorption in violet and blue. Opaque from 0.20μ to 0.275μ. Absorption decreases gradually from 0.275μ to partial transparency at 0.377μ, and then increases to a maximum at 0.42μ. Absorption ends at 0.485μ. No visible absorption from 0.49μ to 0.63μ.

120. Alizarine Blue S. (B.) Sodium bisulphite compound of dioxy-anthraquinone-β-quinoline.
Fig. 28, pl. 7; No. 563, S. & J.
Chocolate-brown powder. In solution yellowish brown, brown.
7.37 g. per liter.
Angle 58.5′. Depth 0 to 0.54 mm.
Absorption in violet, blue, and green. The absorption extends into the ultra-violet. Absorption from 0.20μ to 0.33μ is almost complete save a slight weakening around 0.285μ. Intense maximum at 0.315μ. Absorption decreases abruptly from beyond 0.33μ to transparency at 0.36μ. The transparent region is from 0.36μ to about 0.385μ. A pair of wide bands absorbs from 0.385μ to 0.543μ. Their maxima are at 0.44μ and 0.517μ. The intervening minimum of absorption is at 0.48μ. Transparent from 0.543μ to 0.63μ. The ultra-violet bands remain very intense even when dilution causes the visible bands to disappear.

121. Neutral Red. (D. H.) Hydrochloride of dimethyldiamido-toluphenazine.
Similar to fig 54, pl. 14; No. 580, S. & J.
Dark-green powder. In solution red, pink.
3 g. per liter.
Angle 39.0′. Depth 0 to 0.36 mm.
Band in blue and blue-green, not sharp at edges. Very similar absorption to that of solution No. 122. Slight transparency at 0.23μ. A band lies between this point and 0.31μ where transparency begins to be complete. An absorption band extends from 0.468μ to 0.545μ with its maximum at 0.507μ. Transparent from 0.545μ to 0.63μ.

122. Phenosafranine. Diamidophenyl-phenazonium chloride.
Fig. 54, pl. 14; No. 583, S. & J.
Green, glistening crystals. In solution clear red, pink.
2.5 g. per liter.
Angle 39.0′. Depth 0 to 0.36 mm.

122. Phenosafranine—*Continued.*

Band in blue and blue-green, not sharp at edges. The ultra-violet band ends about 0.31μ. The visible band absorbs from 0.45μ to 0.545μ with maximum at 0.50μ. Transparent from 0.545μ to 0.63μ.

123. Safranine. (B.) Mixture of diamido-phenyl- and tolyl-tolazonium chlorides.

Similar to fig. 54, pl. 14; No. 584, S. & J.

Reddish-brown powder. In solution red, red.

5 g. per liter.

Angle 27.3′. Depth 0 to 0.25 mm.

Definite absorption in blue-green and green. Similar absorption to that of solution No. 122. Strong absorption from 0.20μ to 0.28μ. Rapid decrease in absorption from 0.28μ to 0.32μ. Absorption begins again at 0.44μ, increases to a maximum near 0.495μ, and then decreases to transparency at 0.555μ. Transparent from 0.555μ to 0.63μ.

124. Heliotrope 2 B. (A.) Dimethyldiamido-xylyl-xylophen-azonium chloride.

Fig. 68, pl. 17; No. 590, S. & J.

Grayish-green powder. In solution reddish violet, reddish violet.

7 g. per liter.

Angle 27.3′. Depth 0 to 0.25 mm.

Strong band in yellow and orange. Transparent to deep red. Strong absorption from 0.48μ to 0.605μ with maximum near 0.535μ. Increasing transparency from 0.605μ into the red.

125. Rosolane O. (M.) Phenyldiamido-phenyl-toluphen-azonium chloride.

Similar to fig. 19, pl. 5; No. 591, S. & J.

Olive-green powder. In solution deep violet, faint, reddish violet.

9 g. per liter (warmed and filtered).

Angle 35.1′. Depth 0 to 0.32 mm.

Weak band in the green broadening out into general absorption on the red side. Somewhat similar absorption to that of solution No. 106. The visible band is definitely V-shaped and slopes more at its less refrangible side than at its blue boundary. Slight transparency at 0.23μ is followed by a maximum of absorption near 0.27μ. Absorption ceases about 0.33μ and begins again at 0.495μ. The latter band has its maximum at 0.525μ and fades into general absorption around 0.563μ. The end of the positive slants appreciably, thus emphasizing the general absorption in the orange. Transparent to the red.

126. Nigrosine, soluble. (A.) Sodium salts of sulphonic acids of spirit nigrosines.

Somewhat like figs. 20 and 21 of pls. 5 and 6, respectively; No. 602, S. & J.

Coal-black, glistening lumps. In solution blackish blue, dull blue.

Saturated.

Angle 31.2′. Depth 0 to 0.29 mm.

Very indefinite absorption in the yellow and orange. The ultra-violet absorption is like that of solution No. 127, while the visible absorption is similar to that of solution No. 46. However, the band at 0.528μ is incomparably weaker than the corresponding band of solution No. 46. Absorption rather strong between 0.20μ and 0.30μ. From 0.30μ the absorption decreases to transparency at 0.35μ. Different photographic emulsions show a weak band about 0.528μ. Absorption continues general into the red, as is shown by the appreciable slant of the end of the negative.

127. Naphthalene Red. Mixture of amido-naphthyl-naphthazonium chloride and diamido-naphthyl-naphthazonium chloride.

Fig. 20, pl. 5; No. 614, S. & J.

Dark-brown powder. In solution red, faint pink.

Saturated (heated).

Angle 31.2′. Depth 0 to 0.29 mm.

Hazy-edged absorption band in the green. The visible band extends from 0.488μ to 0.545μ with its maximum at 0.517μ. The slight inclination at the red end of the negative shows appreciable decrease of transparency, between 0.545μ and 0.63μ, with increase of thickness of absorbing layer.

128. Alizarine Green B. (D.) Dioxy-naphthazoxonium sulphonate.

No. 647, S. & J.*

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* Spectrogram too indefinite for reproduction.

128. Alizarine Green B—*Continued.*

Dark-green powder. In solution dark green, light green.

20 g. per liter (heated and filtered).

Angle 31.2′. Depth 0.26 to 0.55 mm.

General absorption in the violet, orange, and red. The minimum of absorption lies in the green, of course. When the depth was 0 to 0.29 mm., the ultra-violet absorption ended at about 0.355μ. There is no sharp band in the ultra-violet, but simply one-sided absorption, decreasing from 0.20μ towards the longer wave-lengths.

129. Columbia Yellow. (A.) Oxidation products of dehydrothiotoluidine-sulphonic acid or of the latter and primuline together.

Fig. 13, pl. 3; No. 663, S. & J.

Brownish-yellow powder. In solution yellow, light yellow.

10 g. per liter (warmed).

Angle 31.2′. Depth 0 to 0.29 mm.

Weak-edged absorption in violet. Uniform absorption from 0.20μ to about 0.36μ. Then a steady decrease in absorption to 0.45μ. Complete transparency from 0.45μ to 0.63μ.

130. Quinoline Blue. (G.) No. 664, S. & J.

Glistening, green crystals. In solution delicate violet, pink.

Saturated (heated).

Angle 0°. Depth 134 mm.

Only the blue and blue-green faintly transmitted. (No photograph was taken for the saturated solution.) When 44 cc. of the saturated solution was diluted to 90 cc. a strong band appeared in the yellow and orange, while the red was freely transmitted. The ultra-violet absorption extends to 0.40μ and fades out about 0.42μ. Weak absorption begins at 0.48μ and increases to opacity near 0.55μ. Complete absorption continues to about 0.607μ. Absorption decreases rapidly from 0.607μ to 0.63μ. When 44 cc. of the saturated solution was made up to 112 cc. and a column 15 cm. was used, no definite band could be seen, but only a faint weakening of the orange. The ultra-violet absorption was, however, complete as far as 0.374μ and vanished near 0.395μ.

131. Quinoline Yellow, soluble in water. (A.) Quinoline Yellow O. (M.) Sodium salt of the sulphonic acid (chiefly disulphonic acid) of quinophthalone.

Fig. 44, pl. 11; No. 667, S. & J.

Yellow powder. In solution lemon yellow, faint yellow.

17.5 g. per liter.

Angle 27.3′. Depth 0 to 0.25 mm.

Strong band in extreme violet. One-sided absorption in ultra-violet ends about 0.33μ. Transparent region from 0.33μ to 0.345μ. A band commences at 0.345μ, has its maximum at 0.40μ, and joins a narrow, companion band at 0.438μ. Maximum of little band is about 0.448μ and complete transparency extends from 0.453μ to 0.63μ. For a 5-strip negative the absorption only advanced to 0.463μ.

132. Indigo Carmine, dry. Sodium salt of indigotine disulphonic acid.

No. 692, S. & J.

Deep-violet powder. In solution blue, blue.

10 g. per liter.

Angle 23.4′. Depth 0 to 0.21 mm.

Transmits green and yellow. Moderately intense band in the orange-red, beginning about 0.555μ and increasing beyond 0.63μ. General absorption from 0.20μ to complete transparency at 0.37μ.

133. Acid Magenta S. (A.) Similar to fig. 53, pl. 14.

Dark-green powder. In solution bluish red, bluish pink.

10 g. per liter.

Angle 15.6′. Depth 0 to 0.14 mm.

Strong band in the green with abrupt edges. Violet, indigo, orange, and red are transmitted. Similar absorption to that of solution No. 95. Partial transparency in the vicinity of 0.265μ. A comparatively weak, narrow band has its maximum near 0.295μ, and absorption ceases about 0.325μ. An intense region of absorption begins at 0.46μ, has its maximum at 0.535μ, and ends at 0.58μ. Therefore this region slants more at the violet border than at the orange side. The band is very smooth and round.

134. Alizarine Orange. (Powder 80 per cent.) (M.)
Fig. 24, pl. 6.
Very deep-purple powder. In solution dark red, red.
5.83 g. per liter (warmed and filtered).
Angle 27.3′. Depth 0 to 0.25 mm.
Weak, shadowy band in the green-yellow, gradually fading away in the red. The ultra-violet absorption is intense and one-sided, and ceases about 0.39μ. The visible band begins near 0.48μ, has its maximum at 0.525μ, and fades into general absorption about 0.56μ. The end of the negative slopes to an unusual extent, showing that the general absorption is relatively strong.

135. Alizarine Red No. 1. 40 per cent. (M.)
Similar to fig. 11, pl. 3.
Yellow paste. In solution deep yellow, yellow.
Saturated.
Angle 31.2′. Depth 0 to 0.29 mm.
Absorption very weak, general, and indefinite. A few mm. of the solution show complete opacity. The solution looks like a milky but yellow emulsion. The ultra-violet absorption is relatively weak, is quite general, and fades away to semi-transparency near 0.4μ. A slight weakening of transparency around 0.52μ may be the fault of the sensitized film. The end of the negative slopes quite enough to indicate continued absorption in the orange. No visible absorption in the red.

136. Aurophosphine 4 G. (A.)
Fig. 45, pl. 12.
Reddish-brown powder. In solution clear yellow, yellow.
11.67 g. per liter.
Angle 27.3′. Depth 0 to 0.25 mm.
Strong absorption in the violet and indigo. Absorption decreases from 0.20μ to a maximum at 0.245μ and then follows a rounded curve to transparency at 0.305μ. Unusual transparency from 0.305μ to 0.390μ. As for *p*-nitrosodimethyl aniline, so here, all the strong lines between 0.324μ and 0.363μ are transmitted with almost no decrease of intensity. A V-shaped band absorbs from 0.390μ to 0.470μ with its maximum at 0.430μ. Transparent from 0.470μ to 0.63μ.

136. Aurophosphine 4 G—*Continued.*

137. Brilliant Croceïne, blue shade. (M.)
Similar to fig. 52, pl. 13.
Bright-red powder. In solution red, salmon pink.
7 g. per liter.
Angle 25.4′. Depth 0 to 0.23 mm.
Strong band in blue-green and green. It slopes more on the blue than on the yellow border. Similar absorption to that of solutions Nos. 21 and 43. Absorption decreases from 0.20μ to a region of partial transparency in the vicinity of 0.29μ. A symmetrical, hazy band absorbs from about 0.305μ to 0.385μ. A strong band absorbs from 0.458μ to 0.558μ with its maximum near 0.518μ. Transparent to the orange and red.

138. Brilliant Purpurine 10 B. (A.)
Similar to fig. 21, pl. 6.
Grayish-violet powder. In solution red, bluish red.
5.83 g. per liter (warmed).
Angle 23.4′. Depth 0 to 0.21 mm.
Shadowy band in the green with general absorption in the yellow and orange. Similar absorption to that of solution No. 46. Absorption decreases from 0.20μ to a semi-transparent region around 0.29μ. A symmetrical, hazy band absorbs from about 0.30μ to 0.38μ. Another band begins near 0.48μ, has its maximum at 0.52μ, and ends at 0.555μ. The end of the negative slants at an angle of about 30°, thus emphasizing the strong, general absorption in the yellow-orange.

139. Carthamin.
Similar to fig. 13, pl. 3.
Reddish-brown plates. In solution brownish red, faint brown.
Concentrated.
Angle 31.2′. Depth 0 to 0.29 mm.
One-sided absorption in the ultra-violet. Absorption similar to that of solution No. 129. Absorption was strong from 0.20μ to about 0.34μ. From this wave-length on, the absorption curve is round and slopes to 0.445μ at the edge of the spectrogram farthest from the comparison spectrum. Transparent from 0.445μ to beyond 0.63μ. The slope

139. Carthamin—*Continued.*

of the blue side of the band is the same as the corresponding region of solution No. 119.

140. Columbia Fast Scarlet 4 B. (A.)

Similar to fig. 26, pl. 7.

Red powder. In solution yellowish red, pink.

5.83 g. per liter.

Angle 27.3′. Depth 0 to 0.25 mm.

Broad absorption in the blue, blue-green, and green. The contour is somewhat hazy. The spectrum is almost identical with fig. 26 for solution No. 69, therefore the wave-lengths are not repeated here.

141. Dianil Orange G. (M.)

Fig. 34, pl. 9.

Brick-red, glistening powder. In solution yellowish red, yellow.

11.67 g. per liter.

Angle 23.4′. Depth 0 to 0.21 mm.

Uniform absorption from the blue border of the green to the extreme violet and beyond. Except for slight, wavy regions, the absorption decreases almost linearly from 0.20μ to about 0.53μ. At this point it ends rather abruptly. Transparent from 0.53μ to 0.63μ.

142. Fluoresceïne. Tetraoxypthalophenone anhydride $C_{20}H_{12}O_5+H_2O$.

Cinnabar-red powder. In solution faint yellow.

Saturated (boiled).

Angle 0°. Depth 3.7 mm.

Weak band in blue and blue-green. Intense, green fluorescence. Absorption complete from 0.20μ to 0.25μ and then decreases to transparency at 0.29μ. Weak absorption extends from about 0.46μ to 0.505μ.

143. Guinea Carmine B. (A.)

Similar to fig. 20, pl. 5.

Brown powder. In solution red, pink.

Saturated.

Angle 2° 17′. Depth 0.26 to 1.51 mm.

Absorption in green. No lines transmitted between 0.20μ and 0.273μ. (Non-zero depth of liquid.) Weak absorption from 0.273μ to 0.33μ. Transparent from 0.33μ to 0.493μ. A hazy-edged band extends from 0.493μ to 0.548μ with its maximum at 0.521μ. Transparent from 0.548μ to 0.63μ. The spectrogram for solution No. 143 does not slant at the red limit, whereas that for solution No. 127 does.

144. Orceïn.

Suggested by figs. 20 and 21 of pls. 5 and 6, respectively.

Black powder with reddish tinge. In solution deep red, light red.

Saturated (heated and filtered).

Angle 31.2′. Depth 0 to 0.29 mm.

Weak, narrow band in the yellow. The ultra-violet absorption is similar to that of solution No. 127. The absorption in the visible spectrum is somewhat like that of solution No. 46. However, the band in the yellow is very much weaker and narrower for solution No. 144 than for the corresponding band of solution No. 46. Strong absorption from 0.20μ to 0.27μ was followed by a gradual decrease to transparency near 0.33μ. Transparent from 0.33μ to about 0.515μ. A narrow, hazy-edged absorption band, with its maximum at 0.525μ, extended from 0.515μ to 0.542μ approximately. The very marked slant of the end of the spectrogram showed the presence of comparatively intense, general absorption in the orange.

145. Soluble Prussian Blue.

Deep-blue, glistening powder. In solution deep blue, blue.

1.76 g. per liter.

Angle 21.3′. Depth 0 to 0.18 mm.

Strong absorption in the yellow, orange, and red. Almost opaque from 0.20μ to 0.28μ. Absorption decreases very gradually from 0.28μ to 0.38μ. The visible region of absorption begins at 0.505μ and continues to 0.63μ and beyond.

146. Thiogene Brown S. (M.)

Similar to fig. 11, pl. 3.

Bluish-black lumps. In solution dull brown, brown.

Saturated.

Angle 27.3′. Depth 0 to 0.25 mm.

General weakening of all the visible spectrum except the blue and red. The solution smells strongly of hydrogen sulphide. (The odor is not so marked for the dry dye.) Similar absorption to that of solution No. 47. Absorption decreases very gradually from 0.20μ to about

146. Thiogene Brown S—*Continued.*

0.41μ. On both sides of 0.445μ the absorption is at a minimum. A very shadowy band absorbs from 0.49μ to 0.545μ. Its maximum is near 0.523μ. The end of the negative slants a good deal more than that of solution No. 47, and thus points to the absorption in the yellow and orange.

147. Thiogene Orange R. (M.)

Similar to fig. 12, pl. 3.

Brown powder. In solution reddish brown, yellow.

5.83 g. per liter (filtered).

Angle 50.7′. Depth 0 to 0.46 mm.

Weak absorption in the violet. The solution has an unpleasant odor. Its absorption is similar to that of solution No. 30. Absorption is complete from 0.20μ to 0.25μ and then decreases with a long, gentle curve to transparency about 0.445μ. Transparent from 0.445μ to 0.63μ.

MISCELLANEOUS ABSORBING MEDIA.

148. Acetone, Ethyl Alcohol, Methyl Alcohol, and Water.

Fig. 87, pl. 22.

The depth of the cell was 1.41 cm. for each of the liquids studied. Especial care was taken to have the three organic solvents as nearly anhydrous and as pure as possible.

The photographic strip nearest to the comparison spectrum gives the absorption of the column of acetone. The next strip in order corresponds to ethyl alcohol. The third strip pertains to methyl alcohol and the strip nearest to the numbered scale is the photographic record for distilled water.

Acetone absorbed all radiations between 0.20μ and 3282.4 Å. U. and the continuous background as far as 3302.7 Å. U.

The most refrangible spark line transmitted by the ethyl alcohol had the wave-length 2265.1 Å. U. This liquid transmitted all the strong ultra-violet lines, but it absorbed the continuous background from 0.20μ to about 0.275μ.

The methyl alcohol transmitted very faintly the strong cadmium line at 2313.0, but no other radiation of wave-length less than 2502.1. The continuous background in the ultra-violet, on the contrary, was transmitted somewhat more freely by the methyl than by the ethyl alcohol.

The distilled water was perfectly transparent to all the radiations in the region photgraphed.

These results show that even ethyl alcohol is not without sufficient absorption in the remote ultra-violet to make it necessary to take this factor into account when columns two or more cm. long are used.

149. Aesculine.

Fig. 73, pl. 19.

White powder. In solution colorless.

Saturated.

Angle 39.0′. Depth 0 to 0.36 mm.

No visible absorption. Intense, blue fluorescence. Absorption decreases from 0.20μ to semi-transparency about 0.26μ. Partial transparency from 0.26μ to 0.273μ. The band with which the fluorescence is probably associated extends from 0.273μ to 0.363μ with the maximum near 0.32μ. Complete transparency from this band to 0.63μ and beyond.

150. Aluminium Chloride, Calcium Bromide, and Calcium Chloride.

Fig. 88, pl. 22.

The depth of the cell was 1.41 cm. for each of the solutions studied. The photographic strip nearest to the comparison spectrum gives the absorption of the calcium bromide solution. The next strip in order corresponds to the aluminium chloride. The third strip from the comparison spectrum pertains to the calcium salt. The remaining strip shows the lack of absorption possessed by distilled water.

The concentrations of the aluminium chloride, calcium bromide, and calcium chloride solutions were, respectively, 2.75, 4.24, and 4.51 normal. The unit used here is the gram-molecular normal; that is, 1 liter of solution of unit concentration would contain 1 gram molecule of the anhydrous salt.

The aluminium chloride solution transmitted faintly all of the strongest

150. Aluminium Chloride, etc.—*Continued.* lines in the remote ultra-violet, but it absorbed the continuous background from 0.20μ to about 0.288μ.

The calcium bromide solution transmitted nothing between 0.20μ and 2748.7. The intensity of this strong cadmium line was greatly diminished. The continuous background began to be perceptible photographically at about 0.313μ.

The calcium chloride solution transmitted faintly all of the strongest lines in the remote ultra-violet, but it absorbed the continuous background from 0.20μ to about 0.280μ.

Consequently there is no very marked difference between the absorptions exerted by the two chlorides. The bromide, on the other hand, possesses much stronger absorption in the ultra-violet region of the spectrum.

151. Barium Permanganate.

The absorption is identical with that of potassium permanganate solutions, having the same concentration in the MnO_4 ions. *See* No. 179.

152. Calcium Bromide.

See No. 150.

153. Calcium Chloride.

See No. 150.

154. Carborundum and Diamond.*

Fig. 89, pl. 22.

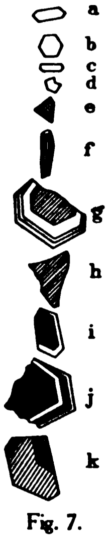

Fig. 7.

Eight crystalline plates of carborundum and three of diamond were fastened to a strip of black paper in such a manner as to bridge across different parts of a long, slit-like opening in the paper. The carborundum plates varied in color from visible transparency to deep blue. The carbons were colorless. The accompanying sketch shows approximately the size, shape, relative positions, and distribution of blue of the plates. *d*, *e*, and *f* denote the diamonds. The paper strip was slid over the slit of the spectrograph, parallel to the length of this opening, and successive exposures were taken.

154. Carborundum and Diamond.—*Cont'd.* The absorption produced by plates *a*, *b*, *c*, *d*, and *e* was first photographed, then the absorption of *f* and *g*, next that of *h* and *i*, and lastly, that of *j* and *k*. The spark and glower exposures were 75 sec. and 60 sec., respectively.

Plate *a* was uniformly colored a blue of moderate intensity. Its absorption is shown by the photographic strip, the outer boundaries of which are numbered 1 and 2. In cases where the crystals were not in contact the light passed through between them and produced narrow comparison spectra; for example, the strip between Nos. 2 and 3.

Plate *b* was almost colorless with a frosted surface. Thickness 0.036 mm. Its absorption spectrum is the strip between 3 and 4.

Plate *c* had about the same color as plate *a*. Thickness 0.173 mm. Its absorption spectrum is the strip between 5 and 6.

Plate *d* was a smooth, colorless carbon. Thickness 0.191 mm. Its spectrum is between 6 and 7. *c* and *d* were practically in contact. This pair of plates shows how much more transparent to ultra-violet light pure carbon is than a colorless plate of carborundum of comparable thickness. Judging by the *negative* the former transmits no light of wavelength shorter than 2748.7 A. U., whereas the latter absorbs everything shorter than 0.390μ.

Plate *e* had such an irregular surface that the light transmitted by it did not fall upon the sensitized film. Thickness about 0.191 mm. The blank between 10 and 11 is due to translation of the photographic film between the first and second settings.

Plate *f* was a diamond with irregularities running parallel to the slit. Thickness 0.533 mm. Spectrum between 11 and 12.

Plate *g* was a deeper blue than any of the above-mentioned crystals in the pentagon nearer plate *f*. The wide border, extending around four sides of the blue area, was practically colorless. Thickness 0.602 mm.

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* Kindly loaned by Mr. L. E. Jewell.

154. Carborundum and Diamond—*Cont'd.*

Spectrum between 13 and 14. *f* and *g* contrast diamond. colorless carborundum, and blue carborundum with one another. The blank from 14 to 15 marks the second setting of the film.

Plate *h* had a delicate, uniform, blue tint. Thickness 0.064 mm. Spectrum between 16 and 17.

Plate *i* was a deeper blue than any of the preceding crystals. Thickness 0.345 mm. Spectrum between 18 and 19. The blank from 19 to 20 corresponds to the third setting of the photographic film.

The center of plate *j* was as deep in color as the middle of *i* and it was also the thickest plate studied. Thickness 0.693 mm. Spectrum between 20 and 21.

Plate *k* was of a delicate blue color of a slightly deeper hue than plates *b* and *c*, except in the corner nearer *j*. In the latter place it had about the same tint as plate *h*. Thickness 0.097 mm. Spectrum between 22 and 23.

155. Chromium Chloride.

Fig. 79, pl. 20.

In solution very dark green, green.

Saturated.

Angle 50.7′. Depth, from nearly 0 to 0.46 mm.

Strong absorption in the violet, blue, orange, and red.

Absorption was complete from 0.20μ to 0.303μ. The boundary of the ultra-violet band curved around from 0.303μ to 0.328μ as the thickness of absorbing layer increased from its least to its greatest value. Semi-transparency from 0.328μ to 0.380μ. A wide, round band, with its maximum near 0.438μ, absorbed from 0.380μ to 0.498μ. This is followed by fairly complete transmission from 0.498μ to 0.555μ. The orange and red region of absorption commenced at about 0.555μ.

156. Cobalt Chloride.

Fig. 78, pl. 20.

In solution red, rose-pink.

351.9 g. of anhydrous salt per liter (2.71 normal).

156. Cobalt Chloride—*Continued.*

Angle 58.5′. Depth 0.53 to 1.07 mm.

One absorption band in the blue-green and another in the deep red.* Absorption was complete from 0.20μ to about 0.248μ. The solution was quite transparent from 0.25μ to about 0.495μ. An absorption band, with its maximum near 0.520μ, extended from 0.497μ to 0.542μ. Transparent from the boundary of this band as far as the deep red.

157. Cobalt Chloride and Aluminium Chloride.

Fig. 95, pl. 24.

The plane-parallel cell was kept at the constant depth of 1.41 cm.

The successive solutions were made up in the following manner: First, a chosen volume of the mother-solution of cobalt chloride was run from a burette or pipette into a measuring flask. Next, a certain amount of the mother-solution of aluminium chloride was run into the same flask and mixed with the solution of the cobalt salt. Finally, distilled water was added to the mixture until the resulting solution filled up the measuring flask to its calibration mark. Of course, all the usual precautions necessary to avoid errors due to changes in volume on mixing and to lack of homogeneity were taken. Each solution of the series was made up to the same volume and contained the same amount of cobalt chloride. On the other hand, the mass of the dehydrating agent present changed from one solution to the next.

The photographic strips nearest to the numbered scale and to the comparison spectrum correspond, respectively, to the solutions which contained the least and greatest amounts of the aluminium salt. The intervening strips succeed one another in the order of increasing percentages of aluminium chloride. The constant concentration of the cobalt chloride in the solutions was 0.271 normal. The concentrations of the aluminium chloride in the several

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*For exhaustive details *see* "Hydrates in Aqueous Solution," etc. Harry C. Jones, Publication No. 60 of the Carnegie Institution of Washington.

157. Cobalt Chloride, etc.—*Continued.*

solutions of the series were 0.000, 1.118, 1.394, 1.676, 1.781, 1.887, 2.096, and 2.459 normal.

The solution which contained no dehydrating agent only absorbed the continuous background from 0.20μ to 0.231μ. The band in the blue-green extended from 0.503μ to about 0.530μ.

The solution of concentration 2.096, in the aluminium chloride, absorbed the continuous background from 0.20μ to 0.288μ. The band in the blue-green extended from 0.485μ to 0.555μ.

The absorption in the yellow and orange is brought out clearly by the photographic strip adjacent to the comparison spectrum. The changes which the bands in the orange and red undergo when the amount of dehydrating agent in the solutions is increased are pronounced and interesting, but they are too complicated to admit of discussion in this place.* Similar changes are brought about by other dehydrating agents, such as calcium chloride, for example.

Figure 95 illustrates the fact that the absorption bands of a colored salt, so-called, can be widened by the addition of suitable colorless salts as well as by simple increase in concentration.

158. Cobalt Chloride in Acetone.

Fig. 90, pl. 23, and fig. 94, pl. 24.

Fig. 90 shows the changes in the positions of the centers of the regions of absorption and transmission of cobalt chloride produced by varying the solvent. The depth of the cell was 2.40 cm. Counting from the comparison spectrum towards the opposite side of the spectrogram, the four photographic strips correspond to solutions of anhydrous cobalt chloride in water, in absolute methyl alcohol, in absolute ethyl alcohol, and in anhydrous acetone, respectively.

The aqueous solution was rosy red. The methyl solution was purple. The color of the ethyl solution was blue with a slight reddish tinge. The solution in acetone was blue with a slight greenish tinge. The concentrations of the solutions, in the order named, were, respectively, 0.325, 0.099, 0.097, and 0.010 normal.

The aqueous solution absorbed practically all radiations from 0.20μ to 0.275μ. The blue-green band absorbed the region between 0.45μ and 0.55μ.

The solution having methyl alcohol for solvent absorbed all of the ultraviolet from 0.20μ to near 0.39μ. It then transmitted from 0.39μ to 0.495μ. The next absorption band extended from 0.495μ to 0.56μ. The faintness of the associated photographic strip shows the presence of appreciable absorption in the yellow.

Both the ultra-violet absorption and the adjoining region of transmission were very nearly the same for the solution in ethyl alcohol as for that in methyl alcohol. On the contrary, the third strip gives no indication of return to transparency in the yellow of the band which absorbed all of the green.

The acetone solution transmitted the region between about 0.38μ and 0.56μ, but absorbed all the other radiations which could affect the Seed film.

The phenomena in the visible spectrum were brought out very clearly by photographing with a Cramer "Trichromatic" plate. The depth of the cell was decreased to 2.00 cm.

The aqueous solution transmitted from beyond the shorter wavelength end of the plate to 0.46μ and again from 0.543μ to beyond 0.625μ at the other end of the plate.

The solution in methyl alcohol transmitted from 0.387μ to 0.495μ and again from 0.548μ to beyond 0.625μ. The intensity of the transmitted light, in the yellow and orange, however, was not as great for the methyl as for the aqueous solution.

The solution in ethyl alcohol only transmitted from 0.385μ to 0.497μ.

The solution in acetone only transmitted from 0.373μ to 0.560μ.

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*For exhaustive details *see* "Hydrates in Aqueous Solution," etc. Harry C. Jones, Publication No. 60 of the Carnegie Institution of Washington.

158. Cobalt Chloride in Acetone—*Cont'd.*

It is thus seen that the photographic center of the band of absorption in the green was displaced by about 200 Ångström units as the solvent was changed from water to methyl alcohol. A still greater displacement was produced by changing from the one alcohol to the other, the concentrations of the two solutions being very nearly equal.

The empirical data given above serve to illustrate* the general fact that the position and character of a given region of absorption or of transmission of a chosen colored salt can be varied, in general, over wide ranges by suitable changes in the solvent used.

Fig. 94 shows the way in which the limits of absorption change when water is added to solutions of anhydrous cobalt chloride dissolved in absolute acetone. The depth of the cell was 2 cm. The solutions were made up in the following manner: A certain arbitrary volume of water was poured into a measuring flask and then the flask was filled up to its calibration mark by running into it from a burette the requisite amount of a mother-solution composed of anhydrous cobalt chloride and absolute acetone. When water is gradually added to such a mother-solution the resulting liquid changes by degrees from deep blue through light blue and then through an almost colorless condition to faint pink.

The percentages by volume of the water in the solutions under consideration were, 0, 2, 4, 6, 8, 10, and 12. The concentration of the mother-solution was 0.015 normal.

The photographic strip nearest to the comparison spectrum corresponds to the solution which was anhydrous. The next strip pertains to the solution which contained 2 per cent of water, and so on, across the entire spectrogram. The mother-solution absorbed completely all radiations between 0.20μ and 0.333μ. The continuous background was very much weakened as far as about 0.361μ. The solution transmitted freely from this wave-length to near 0.552μ. A strong absorption band commenced at 0.552μ and extended into the red.

The photographic strip pertaining to the solution which contained the smallest measured amount of water transmitted from 0.333μ to about 0.566μ. The change in absorption due to the addition of water to the anhydrous mother-solution is, therefore, more noticeable in the ultra-violet than in the yellow. The photographic boundary of the ultra-violet absorption band changed but little, as the percentage of water present in the solutions increased from 2 to 12, and this is due to the intense ultra-violet absorption of the pure acetone. (*See* No. 148.) On the other hand, acetone possesses no absorption band in the visible spectrum, and hence the limits of transmission in the green and yellow, as shown by the several strips of the spectrogram, represent correctly the changes in absorption consequent upon the addition of successive increments of water.

159. Cobalt Chloride in Ethyl Alcohol.
See No. 158.

160. Cobalt Chloride in Methyl Alcohol.
See No. 158.

161. Cobalt Glass.
Fig. 85, pl. 21.

A plane-parallel sheet of ordinary blue cobalt-glass was ground to the form of a wedge and then polished. A prism of colorless glass was attached at the sides to the cobalt prism with its refracting edge parallel to that of the colored glass. The two wedges were in contact over their hypothenuse planes, and hence the outer plane surfaces were nearly parallel. The object in using the colorless glass wedge was, obviously, to correct for the dispersion of the cobalt-glass prism. The lack of agreement between the contiguous edges of the two photographic strips shows that the angle of the colorless prism ought to have been at least twice as large as that of the blue prism. The angle of the cobalt-

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*See also No. 165.

161. Cobalt Glass—*Continued.*

glass wedge was approximately 9°. The compound system absorbed all the ultra-violet from 0.20μ to 0.325μ. The boundary of the ultra-violet band does not curve or slant very much with reference to the long axis of the spectrogram because of the absorption of the colorless glass in this region of the spectrum. The cobalt-glass transmits from about 0.327μ to 0.497μ. Beginning at 0.497μ a region of absorption extends into the red. The most refrangible band in this region has its maximum near 0.52μ. The minimum of absorption between the band just mentioned and the less refrangible, neighboring band is at wavelength 0.560μ. The band in the orange extended into the red beyond the field of view of the spectrograph. These results were tested by using a red-sensitive photographic plate.

162. Cobalt Sulphate.

Similar to fig. 78, pl. 20.

Reddish crystals. In solution red, salmon pink.

Saturated.

Angle about 6°. Depth 0 to about 3.2 mm.

Rather weak absorption in the blue-green. All of the strongest ultra-violet lines were transmitted. The continuous background was absorbed from 0.20μ to about 0.255μ. The band in the blue-green extended from 0.505μ to 0.525μ with its center near 0.515μ.

163. Copper Chloride.

Fig. 77, pl. 20.

Dark-green crystals. In solution dark green, yellowish green.

534.7 g. of anhydrous salt per liter (3.98 normal).

Angle 19.5′. Depth nearly 0 to 0.18 mm.

Intense absorption in the red. The solution was remarkable for its strong absorption of the ultra-violet radiations. Absorption was complete from 0.20μ to 0.32μ at the thinnest part of the wedge. The end of this band curved around from 0.32μ to 0.40μ. Transmission was complete from about 0.40μ to the orange.

164. Copper Chloride and Calcium Chloride.

Fig. 92, pl. 23.

The plane-parallel cell was kept at the constant depth of 1.41 cm. The several solutions were made up as explained under No. 157, which see. The photographic strips nearest to the numbered scale and to the comparison spectrum correspond, respectively, to the solutions which contained the least and greatest amounts of the calcium salt. The intervening strips succeed one another in the order of increasing percentages of calcium chloride. The constant concentration of the copper chloride in the solutions was 0.398 normal. The concentrations of the calcium chloride in the several solutions of the series were 0.000, 0.271, 0.541, 0.812, 1.082, 1.353, 1.624, 1.894, 2.165, 2.435, 2.706, 2.977, 3.247, 3.518, 3.788, and 4.041 normal. The addition of calcium chloride to an aqueous solution of copper chloride changes the color of the latter from clear blue, through green, to yellowish green, due to the presence of an absorption band in the red* and to the encroaching of the ultra-violet band upon the violet and blue.

The solution which contained only copper chloride absorbed all radiations from 0.20μ to about 0.361μ. The solution which contained the greatest amount of the dehydrating agent absorbed all radiations from 0.20μ to about 0.509μ. Hence, the ultra-violet region of absorption widened by about 1480 Ångström units when the concentration of the calcium chloride was increased from 0.000 to 4.041 normal. The spectrogram shows clearly how the successive increments of absorption decreased as the concentration of the calcium salt increased in arithmetical progression. Other dehydrating agents, such as aluminium chloride, for example, produce similar changes in the limits of absorption.

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*For exhaustive details *see* "Hydrates in Aqueous Solution," etc. Harry C. Jones, Publication No. 60 of the Carnegie Institution of Washington.

165. Copper Chloride in Acetone.

Fig. 91, pl. 23, and fig. 93, pl. 24.

Fig. 91 shows the changes in the positions of the ends of the regions of absorption and transmission of copper chloride produced by varying the solvent. The depth of the cell was 1.50 cm. Counting from the comparison spectrum towards the opposite side of the spectrogram, the four photographic strips correspond to solutions of anhydrous copper chloride in absolute acetone, in absolute ethyl alcohol, in anhydrous methyl alcohol, and in water, respectively. The acetone solution was brownish yellow. The ethyl solution was dark green. The color of the methyl solution was yellowish green. The aqueous solution was blue. The concentrations of the solutions, in the order named, were, respectively, 0.022, 0.321, 0.283, and 0.795 normal.

The aqueous solution absorbed all radiations from 0.20μ to 0.387μ and from 0.588μ into the red.

The solution having methyl alcohol for solvent absorbed all of the ultra-violet from 0.20μ to near 0.462μ. It transmitted from 0.462μ to beyond the region of photographic sensibility of the Seed films.

The solution in ethyl alcohol absorbed from 0.20μ to about 0.515μ and again from 0.59μ into the red.

The acetone solution absorbed from 0.20μ to near 0.510μ. It transmitted from 0.510μ to beyond the region of sensibility of the film used.

These results were supplemented by the aid of a Cramer "Trichromatic" plate. A bluish-green, aqueous solution of concentration 1.590 normal was substituted for the one referred to above. The depth of cell and the concentrations of the three remaining solutions were unaltered. This photograph showed that the new aqueous solution transmitted from 0.434μ to 0.588μ, the methyl solution from 0.462μ to beyond 0.625μ, the ethyl solution from 0.513μ to 0.604μ, and the acetone solution from 0.510μ to beyond 0.625μ.

165. Copper Chloride in Acetone—*Cont'd.*

The exposures for the Seed film and the Cramer plate were, respectively, 1.5 and 2 minutes long.

Fig. 93 shows the way in which the limits of absorption change when water is added to solutions of anhydrous copper chloride dissolved in absolute acetone. The depth of the cell was 2 cm. The solutions were made up as explained under No. 158, which see.

The percentages by volume of the water in the solutions under consideration were 0, 1, 2, 3, 4, 6, and 8. The concentration of the mother-solution was 0.022 normal.

The photographic strip nearest to the comparison spectrum corresponds to the solution which was anhydrous. The next strip pertains to the solution which contained 1 per cent of water, etc., across the entire spectrogram. The mother-solution absorbed completely all radiations from 0.20μ to 0.517μ. The next four solutions had a region of transmission the center of which was at 0.436μ. This region was followed by an absorption band whose middle was displaced towards the ultra-violet as the amount of water in the solutions was increased. For the 1 and 2 per cent solutions the center of the absorption band had the approximate wave-lengths 0.478μ and 0.475μ, respectively. The solution which contained 8 per cent of water absorbed all radiations from 0.20μ to about 0.393μ and transmitted from this wave-length to beyond 0.62μ.

166. Copper Chloride in Ethyl Alcohol.

See No. 165.

167. Copper Chloride in Methyl Alcohol.

See No. 165.

168. Diamond.

See No. 154.

169. Erbium Chloride.*

Fig. 101, pl. 26. In solution very faint pink.

Concentrated (filtered).

The solution was poured into a quartz cell, the ends of which were plane and parallel. The cell was successively adjusted to the following

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* A specimen from the collection of the late Prof. Henry A. Rowland.

169. Erbium Chloride—*Continued.*

depths, viz: 0.83, 1.13, 1.43, 1.73, 2.03, 2.33, 2.63, and 2.93 cm. In other words, the thickness of the absorbing layer was increased by 3 mm. between the successive photographic exposures. As has been often remarked by other observers, the solution in question has a very large number of remarkably narrow absorption bands.

For the depth of 0.83 cm. all of the ultra-violet is absorbed from 0.20μ to the cadmium line at 2880.9, while for the depth of 2.93 cm. transmission begins near 0.300μ. The wave-lengths of the maxima of the absorption bands, and the essential characteristics of the bands, as obtained directly from the original negative, are as follows: 0.325μ, 0.350μ, strong with a broad penumbra on both sides; 0.3555μ, faint; 0.3645μ, strong; 0.3662μ, faint companion of the last; 0.3766μ, narrow and faint; 0.3792μ, strong and sharp; 0.3875μ, faint, diffuse band shading off gradually towards the red; 0.405μ, weak and sharp; 0.4075μ, weak; 0.416μ, faint, diffuse band shading off towards the red; 0.419μ, faint; 0.422μ, faint and narrow; 0.427μ, extremely faint and diffuse band; 0.4425μ, faint; 0.450μ, comparatively strong and narrow with a very faint companion at the more refrangible side and with a broad, hazy band near the opposite edge; 0.4675μ, very faint; 0.4725μ, very faint and diffuse; 0.480μ, extremely faint; 0.485μ, weak; 0.4875μ, comparatively strong and narrow; 0.491μ, wide, hazy band shading off towards the red; 0.5186μ, weak and narrow; 0.5205μ, narrow; 0.5235μ, strong and narrow; 0.5365μ, weak and broad; and 0.5413μ, weak with a broad, diffuse companion on the side nearest to the red.

170. Ethyl Alcohol.

See No. 148.

171. Glycerine.

A plane-parallel layer of glycerine 13.5 mm. deep absorbed all light of wave-length less than 0.25μ and it produced a general weakening of the continuous background as far as about 0.33μ. The exposure lasted for 1.5 minutes.

172. Litmus.

Figs. 83 and 84, pl. 21.

In solution blue and red for the neutral (or alkaline) and acid conditions, respectively.

Saturated.

Angle, about 6° for both cases. Depth 0 to 3.2 mm., approximately, for fig. 83.

The absorption of the blue solution is suggested by fig. 83. Absorption was practically complete from 0.20μ to about 0.28μ. From this wave-length the absorption band followed a gentle slope to about 0.42μ for the greatest depth of solution. A region of partial transparency extended from 0.42μ to near 0.496μ. A band of absorption began at 0.496μ and had its maximum approximately at 0.531μ. The spectrogram indicates the existence of intense absorption in the orange and red.

Fig. 84 gives the photographic record obtained with an acid solution of litmus. This solution absorbed the greater part of the ultra-violet region just as the neutral solution did. On the other hand, the acid solution exerted general absorption in the violet and blue, whereas the neutral solution, of the same depth, transmitted the light of these colors. The maximum of the band in the green was at 0.515μ for the red solution. The displacement of this maximum from 0.531μ to 0.515μ was probably exaggerated by the variations of sensibility of the photographic films for radiations of different wave-lengths. Fig. 84 recorded only weak absorption in the yellow-orange. Red was transmitted.

173. Methyl Alcohol.

See No. 148.

174. Neodymium Ammonium Nitrate.

Figs. 96, 97, and 98, pl. 25.

Pink crystals. In solution pink.

Concentrated (filtered).

For fig. 96 the solution was poured into a quartz cell the ends of

174. Neodymium Ammonium Nitrate—*Continued.*

which were plane and parallel. The cell was successively adjusted to the following depths, viz: 0.53, 0.83, 1.13, 1.43, 1.73, 2.03, 2.33, and 2.63 cm. In other words, the thickness of the absorbing layer was increased by 3 mm. between the successive photographic exposures. As has been often remarked by other observers, the solution in question has a large number of unusually narrow absorption bands, some of which are very intense and persistent.

For the depth of 0.53 cm. all of the ultra-violet is absorbed from 0.20μ to the zinc line at 3302.7, while for the depth of 2.63 cm. only very faint transmission obtains in the immediate vicinity of 3407.7 Å. U. The general characteristics of the most intense bands can be readily seen by referring to fig. 96, hence it will suffice to give the approximate wave-lengths of the absorption bands which were recorded by the original negative.

The centers of the bands were at 0.347μ, 0.350μ, 0.355μ, 0.381μ, very faint; 0.418μ, faint; 0.4275μ, sharp; 0.433μ, very faint; 0.4437μ, diffuse; 0.461μ, faint and diffuse; 0.4695μ, 0.4755μ, faint; 0.4823μ, 0.5087μ, 0.5112μ, with a hazy boundary at the less refrangible side; 0.520μ, 0.5225μ, broad and intense; 0.5324μ, faint; 0.5775μ, broad and intense, and 0.5925μ, faint and diffuse.

Fig. 97 shows the absorption of the same solution when placed in the wedge-shaped cell. The angle of the liquid wedge was 1° 18′ and the depth increased linearly from 0.71 mm. to 1.24 mm. Except for the transmission of the strong metallic lines at 2558.0, 2573.1, and 2748.7, the ultra-violet absorption is practically complete as far as 0.3250μ. The negative for fig. 97 recorded very faintly all of the absorption bands given above except the ones at 0.347μ, 0.350μ, 0.381μ, 0.418μ, 0.461, and 0.5324μ.

174. Neodymium Ammonium Nitrate—*Continued.*

The angle of the cell was 39′ for fig. 98, so that the thickness of the absorbing layer varied from about 0 to 0.36 mm. Absorption was complete from 0.20μ to 0.237μ. The boundary of this region of absorption curved around rather abruptly from 0.237μ to 0.250μ as the depth of solution increased from its least to its greatest value. Transmission by the deepest part of the liquid wedge was weakened somewhat from 0.277μ to 0.308μ. Only the intense absorption band at wave-length 5225 Å. U. was recorded by the negative.

175. Nickel Nitrate.

Fig. 81, pl. 21.

Green crystals. In solution green, light green.

Saturated.

Angle, about 6°. Depth 0 to 3.2 mm., approximately.

Strong absorption in the orange and red, also weaker absorption in the extreme violet. The absorption was nearly complete from 0.20μ to about 0.312μ. The end of this region of absorption curved around from 0.312μ to 0.326μ with increasing depth of solution. Unusual transparency from 0.326μ to 0.374μ. A symmetrical absorption band, with its maximum at 0.391μ, extended from 0.374μ to 0.408μ. Transmission was complete from this point as far as the absorption band in the orange. The sloping end of the spectrogram calls attention to absorption in the orange.

176. Nickel Sulphate.

Fig. 82, pl. 21.

Green crystals. In solution green, pale green.

Saturated.

Angle, about 6°. Greatest depth, 3.2 mm., approximately. Absorption in the extreme violet, orange, and red. Using the faint comparison spectrum as a standard of comparison, it becomes evident that the solution was remarkably transparent to the ultra-violet radiations from 0.226μ to about 0.365μ. A symmetrical absorption band,

176. Nickel Sulphate—*Continued.*
with its maximum at 0.391μ, extended from 0.367μ to 0.415μ. Transmission was complete from this point as far as the absorption band in the orange. A comparison of figs. 81 and 82 is very suggestive. Both spectrograms show the same band at wave-length 0.391μ, but the ultra-violet absorption exerted by the nitrate is entirely different from that shown by the sulphate.

177. Picric Acid.
Fig. 86, pl. 22.
Yellow crystals with greenish hue. In solution yellow, pale yellow.
Concentration unknown.
Angle 50.7′. Depth 0 to 0.46 mm.
Hazy band in the violet extending into the ultra-violet. Absorption decreased gradually from 0.20μ to partial transparency at 0.275μ. Semi-transparency from 0.275μ to 0.300μ. A band of absorption, with hazy contour, extended from about 0.300μ to 0.400μ, its maximum being near 0.35μ.

178. Potassium Chromate.
Fig. 80, pl. 20.
Yellow crystals. In solution yellow, faint yellow.
Very dilute.
Angle 50.7′. Change in depth 0.46 mm.
Absorption in the extreme violet extending into the ultra-violet. The most refrangible absorption band only extends from beyond 0.20μ to 0.226μ. The solution is noticeably transparent to all radiations from 2265.1 Å. U. to 2321.2 Å. U., inclusive of these limits. An intense band extends from 0.227μ to 0.300μ. This is followed by a region of almost complete transparency, the middle of which is near 0.316μ. A strong band of absorption extends from 0.332μ to 0.406μ with its maximum at 0.369μ.

179. Potassium Permanganate.
Figs. 74 and 75, pl. 19.
Grayish-brown crystals with violet reflex. In solution deep violet, violet.

179. Potassium Permanganate—*Continued.*
16.67 g. per liter.
Angle 27.3′. Depth 0 to 0.25 mm.
Five distinct bands clearly visible in the green with a very faint companion on the blue side. The central band of the five is a little more intense than its less refrangible neighbor. Light from the spark decomposes the potassium permanganate so rapidly, with the formation of innumerable small bubbles, that the exposures had to be made as follows: 1st. Expose to spark for 25 seconds. 2d. Remove cell from spectrograph and clean away the bubbles. 3d. Replace the cell and make another exposure for 25 seconds, etc., three times for each distinct strip of the spectrogram. The absorption at 0.20μ is weak and decreases to transparency near 0.25μ. Unusual transparency from 0.25μ to 0.29μ. This fact is brought out in a half-dozen spectrograms of the region. A band of absorption extends roughly from 0.29μ to 0.36μ with its maximum at the center. The transparency increases to completeness and continues to 0.483μ. The wave-lengths of the 7 photographic bands are 0.457μ, 0.472μ, 0.488μ, 0.505μ, 0.525μ, 0.545μ, and 0.570μ. (Only 5 bands show on the complete spectrogram.) In decreasing order of intensity the three strongest bands are 0.525μ, 0.505μ and 0.545μ.
An effort was made to detect the 8 bands given by Formánek,* but the conditions were not favorable to recording more than seven bands. Formánek's wave-lengths are "571.0, 547.3 (Hauptstreifen), 525.6, 505.4. 487.0, 470.7, 454.4, and 439.5."
The negative for fig. 75, pl. 19, shows the seven bands. The solution was practically saturated since it contained 50 grams per liter at room temperature. Here the ultra-violet absorption extends as far as 0.39μ. For concentrations from 16.67 to 50 grams per liter, and for

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*See J. Formánek, "Die qualitative Spectralanalyse anorganischer Körper," p. 59.

179. Potassium Permanganate—***Continued.*** the method used, the bands do not shift at all.* Trichromatic plates were used to see if any photographic bands less refrangible than 0.570μ could be recorded. No evidence of the existence of such bands was presented.

180. Praseodymium Ammonium Nitrate. Fig. 100, pl. 26.

Yellowish-green crystals. In solution yellowish green.

Concentrated (filtered).

The solution was poured into a quartz cell, the ends of which were plane and parallel. The cell was successively adjusted to the following depths, viz: 0.73, 1.03, 1.33, 1.63, 1.93, 2.23, 2.53, and 2.83 cm. In other words, the thickness of the absorbing layer was increased by 3 mm. between the successive photographic exposures.

The solution is remarkable for the comparative narrowness and great intensity of its absorption bands. Absorption was complete from 0.20μ to about 0.333μ and 0.343μ, respectively, for the least and greatest depths of solution investigated. The centers of the four intense bands which fell within the region of sensitivity of the Seed emulsion were at wave-lengths 0.4445μ, 0.4685μ, 0.4820μ, and 0.590μ. The least refrangible side of the band at 0.590μ does not appear in fig. 100 because the band came very near the limit of sensibility of the photographic film employed.

181. Sodium Bichromate.

Suggested by fig. 14, pl. 4.

Orange-red crystals. In solution yellow, pale yellow.

Very dilute solution.

Angle 50.7′. Depth 0 to 0.46 mm.

181. Sodium Bichromate—***Continued.***

The spectrogram differs from fig. 14 in having the ultra-violet absorption curve displaced bodily towards the region of the shortest wave-lengths. Absorption was practically complete from 0.20μ to about 0.27μ, for all depths. At the thickest part of the liquid wedge absorption was complete from 0.20μ to 0.40μ, but both the photographic strip adjacent to the comparison spectrum and the one in the middle of the spectrogram recorded a comparatively narrow band of semi-transparency, the center and maximum of which was near 0.318μ. This was followed by a strong, round absorption band whose maximum was at 0.36μ. In other words, there were two round, ultra-violet bands of absorption which coalesced at the wave-length 0.318μ. Transmission was complete from 0.40μ to 0.63μ.

182. Sodium Nitroprussid.

Fig. 76, pl. 19.

Garnet crystals. In solution reddish brown, light brown.

Saturated.

Angle 1° 45′. Depth 0 to 0.96 mm.

Weak absorption in violet. Light from the spark decomposes the solution at the very beginning of illumination so that the method used for photographing the ultra-violet absorption of the permanganates was not applicable. This difficulty was not overcome. Absorption decreases to about 0.38μ, then increases to a weak maximum near 0.396μ, and finally decreases to transparency at 0.428μ. No selective absorption from 0.43μ to 0.62μ.

183. Water. *See* No. 148.

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* *See* H. Kayser, "Handbuch der Spectroscopie," v. iii, p. 415.

Alphabetical List of Absorbing Media.

	Page	No.	Pl.	Fig.
Acetone	46	148	22	87
Acid Brown	27	42	....	....
Acid Green, conc'	35	86	12	46
Acid Magenta S	43	133	....	....
Acid Rosamine A.	38	106	5	19
Aesculine	46	149	19	73
Alizarine Blue S.	41	120	7	28
Alizarine Brown	41	118	....	....
Alizarine Green B.	42	128	....	....
Alizarine Orange. (Powder 80%)	44	134	6	24
Alizarine Red No. 1, 40%	44	135	....	....
Alizarine Red S.	41	119	4	14
Alkali Blue 6 B.	37	97	18	72
Aluminium Chloride	46	150	22	88
Amidonaphtholdisulphonic Acid H	21	1	1	1
Anthracene Red	31	62	....	....
Anthracene Yellow C.	29	53	....	....
Auramine O.	34	82	11	43
Aurantia	21	8	10	39
Aurophosphine 4 G.	44	136	12	45
Azo Acid Yellow	25	31	....	....
Azo Blue	32	72	....	....
Azo Coccine 2 R.	23	14	2	9
Azo Rubine S.	26	35	....	....
Azo Yellow, conc'	25	31	...	....
Barium Permanganate	47	151	19	74 75
Benzoazurine	33	75	....	....
Benzopurpurine B.	31	65	....	....
Benzopurpurine 6 B.	31	64	....	....
Benzopurpurine 10 B.	33	74	6	22
Biebrich Scarlet	28	49	....	....
Bismarck Brown	29	54	2	7
Blue Black	29	52	....	....
Bordeaux B.	23	19	....	....
Brilliant Congo R.	32	67	....	....
Brilliant Croceïne, blue shade	44	137	....	....
Brilliant Orange G.	23	15	8	31
Brilliant Purpurine 10 B.	44	138	....	....
Brilliant Purpurine R.	32	69	7	26
Calcium Bromide	47	152	22	88
Calcium Chloride	47	153	22	88
Carborundum	47	154	22	89
Carthamin	44	139	....	....
China Blue	37	99	18	71
Chromium Chloride	48	155	20	79
Chromotrope 6 B.	22	13	2	8
Chrysamine G.	30	57	....	....
Chrysoïdine	22	12	2	7
Cloth Red G.	28	45	7	25
Cloth Red 3 G A.	28	47	3	11
Cloth Red O.	28	46	6	21
Cobalt Chloride	48	156	20	78
Cobalt Chloride and Aluminium Chloride	48	157	24	95
Cobalt Chloride in Acetone	49	158	23 24	90 94
Cobalt Chloride in Ethyl Alcohol.	50	159	23 24	90 94
Cobalt Chloride in Methyl Alcohol	50	160	23 24	90 94
Cobalt Glass	50	161	21	85
Cobalt Sulphate	51	162	....	....
Coccinine B.	23	20	....	....
Columbia Fast Scarlet 4 B.	45	140	....	....
Columbia Yellow	43	129	3	13
Congo Brown G.	33	77	9	35 36
Congo Brown R.	33	78	9	35
Congo Corinth	32	71	....	....
Congo Corinth G.	30	60	6	23
Congo Orange G.	29	56	9	33
Congo Orange R.	31	63	....	....
Congo Red	30	59	....	....
Congo Rubine	30	61	....	....
Copper Chloride	51	163	20	77
Copper Chloride and Calcium Chloride	51	164	23	92
Copper Chloride in Acetone	52	165	23 24	91 93
Copper Chloride in Ethyl Alcohol.	52	166	23 24	91 93
Copper Chloride in Methyl Alcohol	52	167	23 24	91 93
Coralline Red	37	100	13	49
Cresotine Yellow G.	30	58	....	....
Crystal Ponceau 6 R.	23	18	....	....
Crystal Violet	35	90	....	....
Curcumeïne	25	30	3	12
Curcumine S.	34	81	10	37
Cyanosine	40	113	5	18
Dahlia	35	89	18	69
Deltapurpurine 5 B.	31	66	....	....
Diamine Black B O.	33	73	....	....
Diamine Green B.	33	76	....	....
Diamine Red B.	31	66	....	....
Diamine Red 3 B.	32	68	16	61
Diamond	52	168	22	89
Dianil Orange G.	45	141	9	34
Dianil Yellow R.	27	40	....	....
Emerald Green	34	84	....	....
Emin Red	24	23	8	29
Eosamine B.	24	21	13	52
Eosine à l'alcool	39	109	5	17
Eosine, bluish	40	111	....	....
Eosine, yellowish	39	108	15	58
Erbium Chloride	52	169	26	101
Erika B.	24	22	15	57
Erythrosine	40	112	15	59
Ethyl Alcohol	53	170	22	87
Ethyl Violet	36	91	16	64
Fast Acid Violet A 2 R.	38	105	....	....
Fast Acid Violet B.	38	104	16	63
Fast Brown 3 B.	26	38	....	....
Fast Green O	34	79	....	....
Fast Red A	26	34	7	27
Fast Red, extra	26	36	....	....
Fast Yellow	22	9	....	....
Fluoresceïne	45	142	....	....
Fuchsine	35	87	12	48
Fuchsine S.	36	95	14	53
Galleïne	40	116	....	....
Glycerine	53	171	....	....
Guinea Carmine B.	45	143	....	....
Heliotrope 2 B.	42	124	17	68
Indigo Carmine, dry	43	132	....	....
Janus Green	24	24	....	....

ALPHABETICAL LIST OF ABSORBING MEDIA.—*Continued.*

	Page	No.	Pl.	Fig.
Janus Red B	28	44	....	....
Light Green F S	35	85	....	....
Litmus	53	172	21	83 84
Malachite Green	34	83	....	....
Metanil Yellow	25	32	10	40
Methyl Alcohol	53	173	22	87
Methyl Blue	37	98	....	....
Methyl Eosine	39	110	....	....
Methyl Green	36	94	12	47
Methyl Green OO	36	93	....	....
Methyl Orange III	25	28	11	41
Methyl Violet 6 B	36	92	17	66
Mordant Yellow O.	27	39	....	....
Naphthalene Red	42	127	5	20
β-Naphtholdisulphonic Acid G	21	2	1	2 5
Naphthol Green B..	34	80	3	10
Naphthol Yellow	21	7	11	42
Naphthol Yellow S.	21	7	11	42
Naphthylamine Brown	25	33	....	....
Neodymium Ammonium Nitrate.	53	174	25	96 97 98
Neutral Red	41	121	....	....
New Coccine	26	37	....	....
New Coccine O	26	34	7	27
New Magenta..	35	88	13	50
Nickel Nitrate	54	175	21	81
Nickel Sulphate	54	176	21	82
Night Blue	37	101	....	....
Nigrosine, soluble	42	126	....	....
p-Nitraniline. (Powder, "extra.")	21	3	....	....
o-Nitrobenzaldehyde	21	4	....	....
p-Nitrosodimethylaniline	21	5	1	3
Orange G	22	10	8	30
Orceïn	45	144	....	....
Phenosafranine	41	122	14	54
Phloxine	40	115	15	60
Phosphine	41	117	8	32
Picric Acid	55	177	22	86
Ponceau B O, extra	27	43	....	....
Ponceau 2 G	22	11	2	6
Ponceau 2 R	23	16	....	....
Ponceau 3 R	23	17	14	55
Ponceau 4 R B	28	48	13	51
Ponceau 6 R B	29	51	14	56
Potassium Chromate	55	178	20	80
Potassium Permanganate	55	179	19	74 75
Praseodymium Ammonium Nitrate	56	180	26	100
Quinoline Blue	43	130	....	....
Quinoline Yellow O	43	131	11	44
Quinoline Yellow, sol. in water..	43	131	11	44
Red Violet 5 R S	37	96	16	62
Resorcine (techn. pure)	21	6	1	4
Resorcine Brown	27	41	10	38
Rhodamine B	38	103	17	65
Rosazurine B	32	70	....	....
Rose Bengal	40	114	....	....
Rosolane O	42	125	....	....
Safranine	42	123	....	....
Sodium Bichromate	56	181	....	....
Sodium Nitroprussïd	56	182	19	76
Soluble Prussian Blue	45	145	....	....
Thiogene Brown S	45	146	....	....
Thiogene Orange R	46	147	....	....
Tropæoline O	24	25	....	....
Tropæoline OO	25	29	....	....
Tropæoline OOO No. 1	24	26	....	....
Tropæoline OOO No. 2	25	27	....	....
Uranine	38	107	4	15 16
Vesuvine	29	55	2	7
Victoria Blue 4 R	37	102	18	70
Water	56	183	22	87
Wool Black	29	50	17	67

For convenience in identifying the stronger spark lines, fig. 99, pl. 25, is given.* The numbers on this positive correspond to those preceding the wave-lengths below. The wave-lengths were derived from the two following sources:

"An Introduction to the Study of Spectrum Analysis," by W. M. Watts. Longmans, Green & Co., 1904.

"Measurements of the Wave-lengths of Lines of High Refrangibility in the Spectra of Elementary Substances." Hartley and Adeney. From the Philosophical Transactions of the Royal Society. Part I, 1884.

No.	W.-l.	Radiator.	
1.	2024.2	Zn.	
2.	2060.8 2062.8	Zn.	
3.	2099.0	Zn.	
4.	2138.3	Zn.	
5.	2144.5	Cd.	
6.	2194.7	Cd.	
7.	2239.9	Cd.	
8.	2265.1	Cd.	
9.	2288.1	Cd.	
10.	2306.7	Cd.	
11.	2313.0	Cd.	
12.	2321.2	Cd.	
13.	2329.3	Cd.	
14.	2502.1	Zn.	
15.	2558.0	Zn.	
16.	2573.1	Cd.	
17.	2710.1 2712.6	Air. Zn.	
18.	2748.7	Cd.	
19.	2770.9	Zn.	
20.	2801.0	Zn.	
21.	2837.0	Cd.	
22.	2880.9	Cd.	
23.	2980.8	Cd.	
24.	3007.0	Air.	
25.	3035.9	Zn.	
26.	3072.2 3076.0	Zn.	
27.	3133.3	Cd.	
28.	3250.5	Cd.	
29.	3261.2	Cd.	
30.	3282.4	Zn.	
31.	3302.7	Zn.	
32.	3329.3 3331.5	Air.	
33.	3345.1 3345.5	Zn.	
34.	3407.7	Cd.	
35.	3436.9	Air.	
36.	3466.3 3467.8	Cd.	
37.	3530.0	Cd.	
38.	3610.7 3613.0	Cd.	
39.	3682.6†		
40.	3712.2	Air.	
41.	3726.6	Air.	
42.	3749.8	Air.	
43.	3839.3 3841.7	Air.	
44.	3881.9	Air.	
45.	3919.2	Air.	
46.	3954.8 3956.2	Air.	
47.	3972.5	Air.	
48.	3995.1	Air.	
49.	4041.4	Air.	
50.	4070.0 4072.4 4076.1	Air.	
51.	4119.4	Air.	
52.	4132.8 4133.8	Air.	
53.	4145.9	Air.	
54.	4151.9 4153.6	Air.	
55.	4228.5 4236.7 4241.9	Air.	
56.	4316.2 4318.7	Air.	
57.	4349.5	Air.	
58.	4367.9	Air.	
59.	4415.1	Air.	
60.	4447.1 4447.2	Air.	
61.	4530.1	Air.	
62.	4576.2	Cd.	9_2 ‡
63.	4601.6	Air.	
64.	4607.3	Air.	
65.	4614.0	Air.	
66.	4621.6	Air.	
67.	4626.0	Cd.	11_2
68.	4630.7	Air.	
69.	4641.9 4643.4	Air.	
70.	4649.2	Air.	
71.	4680.4	Zn.	
72.	4722.3	Zn.	
73.	4800.1	Cd.	
74.	4810.7	Zn.	
75.	4912.3	Zn.	
76.	4924.8	Zn.	
77.	5002.7 5005.7	Air.	
78.	5045.7	Air.	
79.	5086.1	Cd.	
80.	5116.0	Zn.	15_2
81.	5146.2	Cd.	16_2
82.	5338.6	Cd.	$—_2$
83.	5354.4	Cd.	$—_2$
84.	5379.3	Cd.	
85.	5497.4	Cd.	18_2
86.	5509.0	Zn.	$—_2$
87.	5541.8	Zn.	19_2
88.	5602.0	Zn.	20_2
89.	5761.8	Cd.	22_2
90.	5961.6	Cd.	23_2
91.	6014.0	Air.	24_2
92.	6035.0	Zn.	$—_2$
93.	6071.8	Zn.	25_2
94.	6144.4	Zn.	26_2
95.	6152.0	Zn.	26_2
96.	6160.4	Cd.	$—_2$
97.	6266.6	Cd.	27_2

The following table facilitates the finding of the numbers, names, etc., of all the substances which have an absorption spectrum more or less similar to that shown by a selected spectrogram. The third column gives the number of every substance referred in the text to the plate and figure of the preceding columns.

Plate.	Fig.	No.
3	11	33, 63, 79, 116, 118, 135, 146.
3	12	31, 147.
3	13	39, 139.
4	14	181.
5	19	19, 61, 105, 125.
5	20	126, 143, 144.
6	21	126, 138, 144.
6	22	71, 72, 73, 75.
6	23	38, 44.
7	25	70.
7	26	59, 64, 65, 66, 67, 70, 140.
8	31	26, 27.
9	36	57, 58.
10	37	25, 40, 53.
10	39	42.
10	40	9, 29.
12	46	83, 84, 85.
12	47	93.
13	51	49.
13	52	37, 43, 137.
14	53	133.
14	54	121, 123.
14	55	16, 18, 20, 35, 36.
14	56	62.
15	58	110, 111.
15	59	114.
16	63	105.
17	66	90.
18	70	101.
18	71	98.
20	78	162.

---

*The negative was not a single exposure. To stand reproduction the extreme ultra-violet was "favored."

†"Doubtful Origin." ‡The subscript 2 denotes the second order of spectrum.

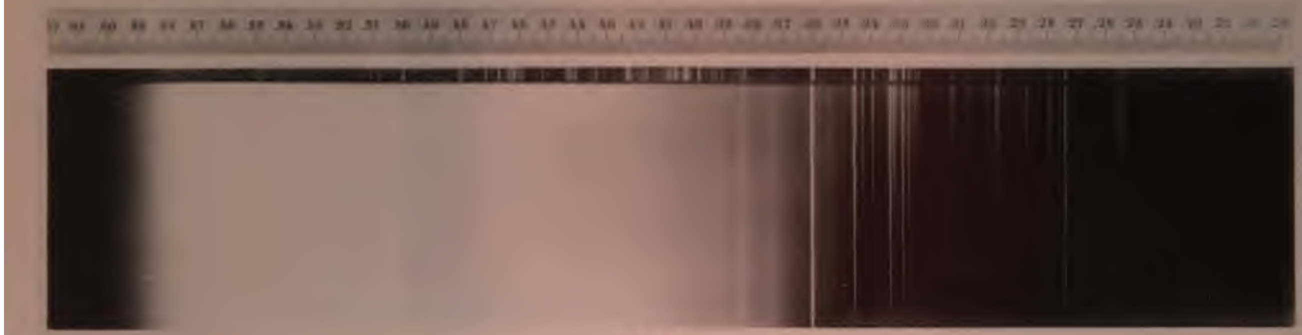


FIG. 1.

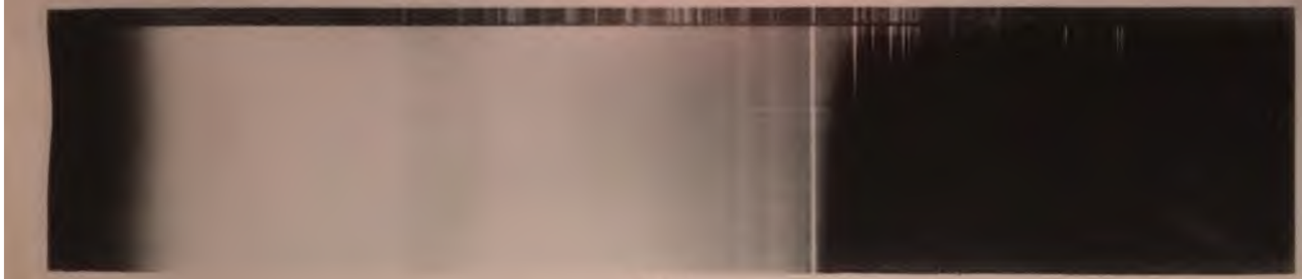


FIG. 2.

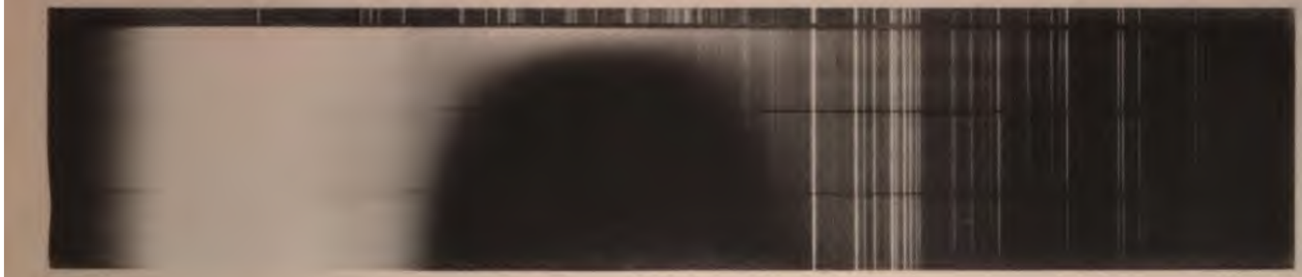


FIG. 3.

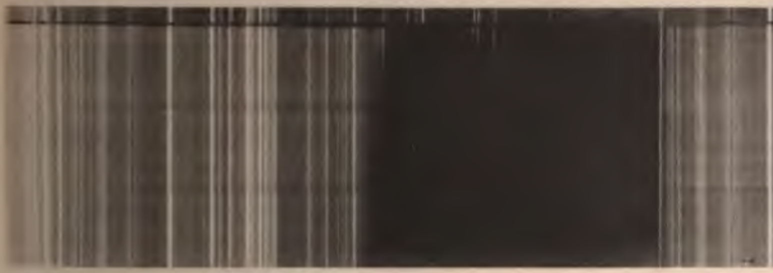


FIG. 4.

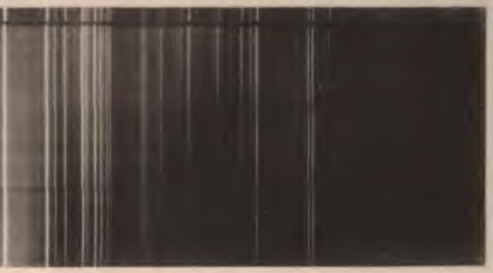


FIG. 5.

FIG. 1. SEE NO. 1. AMIDONAPHTHOLDISULPHONIC ACID H.
FIG. 2. SEE NO. 2. β-NAPHTHOLDISULPHONIC ACID G.
FIG. 3. SEE NO. 5. *p*-NITROSODIMETHYLANILINE.
FIG. 4. SEE NO. 6. RESORCINE.
FIG. 5. SEE NO. 2.

HELIOTYPE CO., BOSTON.

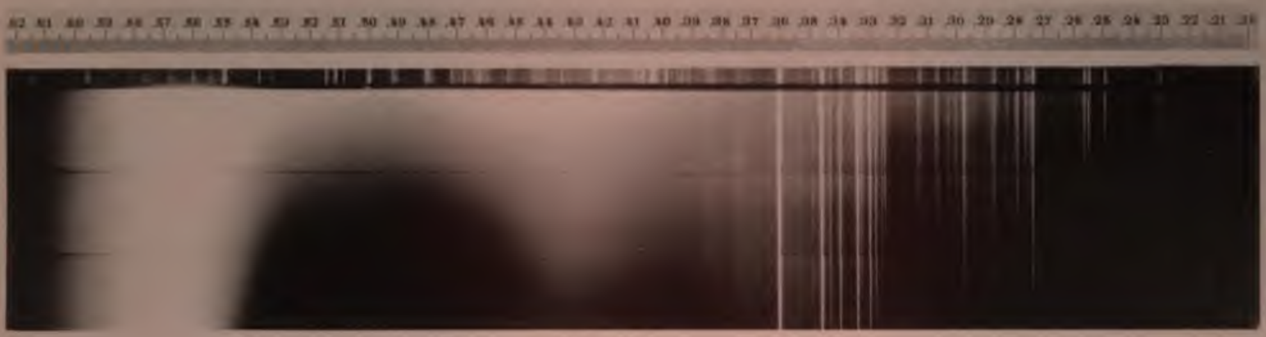


FIG. 6.

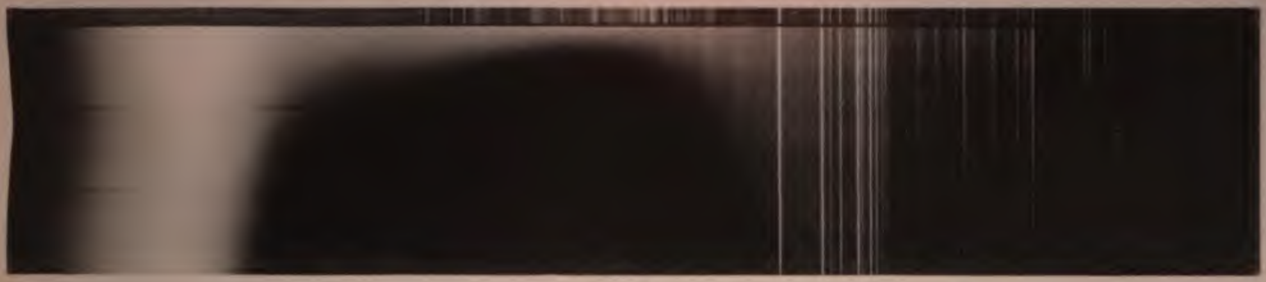


FIG. 7.

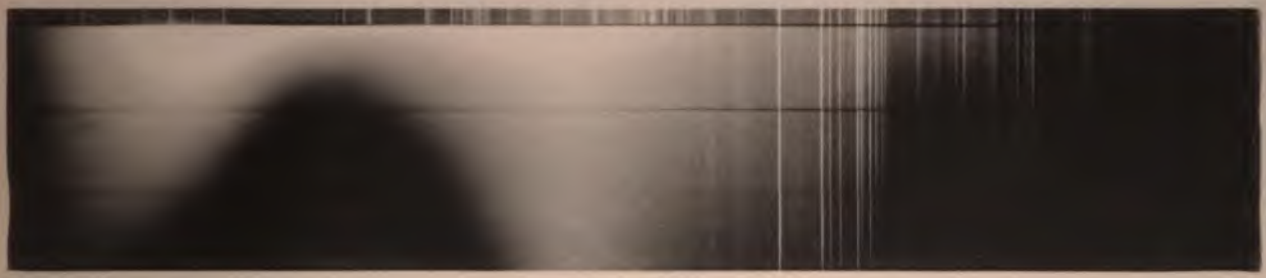


FIG. 8.

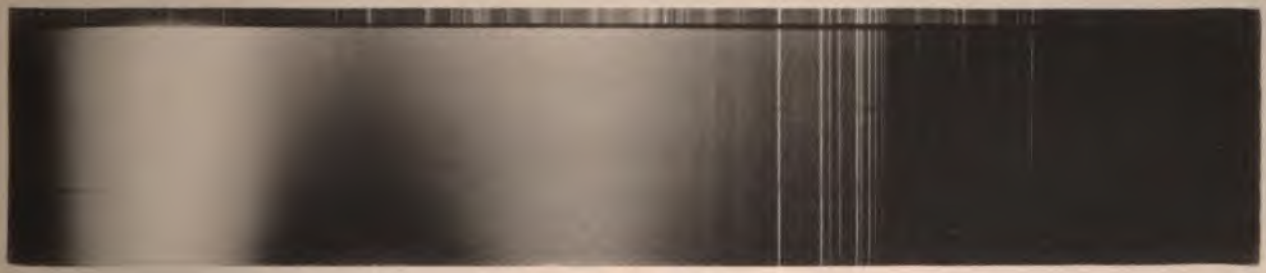


FIG. 9.

FIG. 6. SEE NO. 11. PONCEAU 2 G.
FIG. 7. SEE NO. 12. CHRYSOÏDINE.
FIG. 8. SEE NO. 13. CHROMOTROPE 6 B.
FIG. 9. SEE NO. 14. AZO COCCINE 2 R.

HELIOTYPE CO., BOSTON.

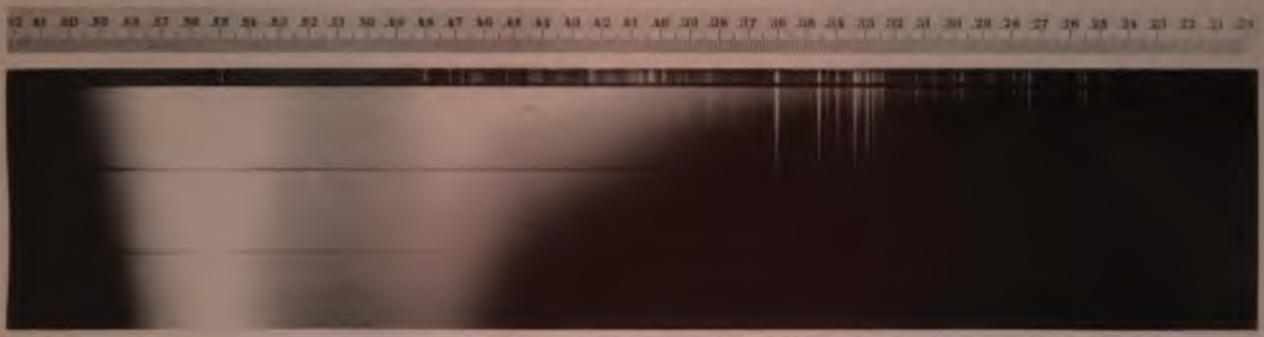


FIG. 10.

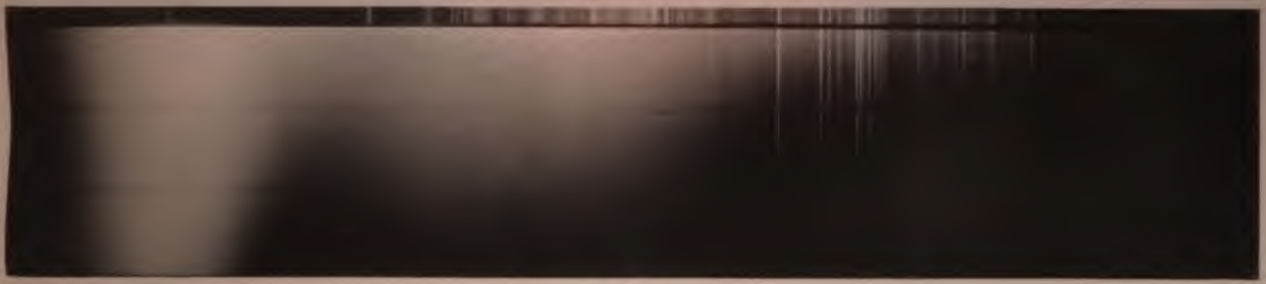


FIG. 11.

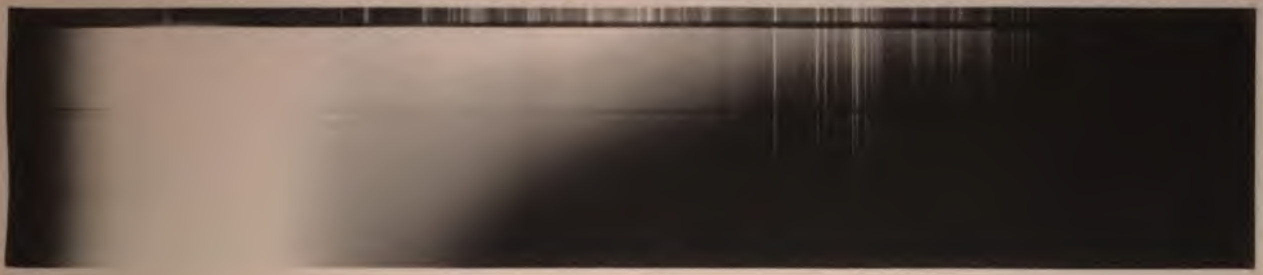


FIG. 12.

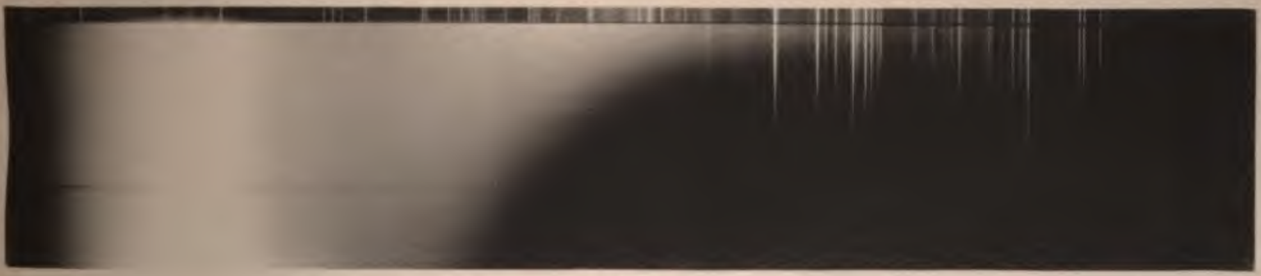


FIG. 13.

FIG. 10. SEE NO. 80. NAPHTHOL GREEN B.
FIG. 11. SEE NO. 47. CLOTH RED 3 G A.
FIG. 12. SEE NO. 30. CURCUMEÏNE.
FIG. 13. SEE NO. 129. COLUMBIA YELLOW.

HELIOTYPE CO., BOSTON.

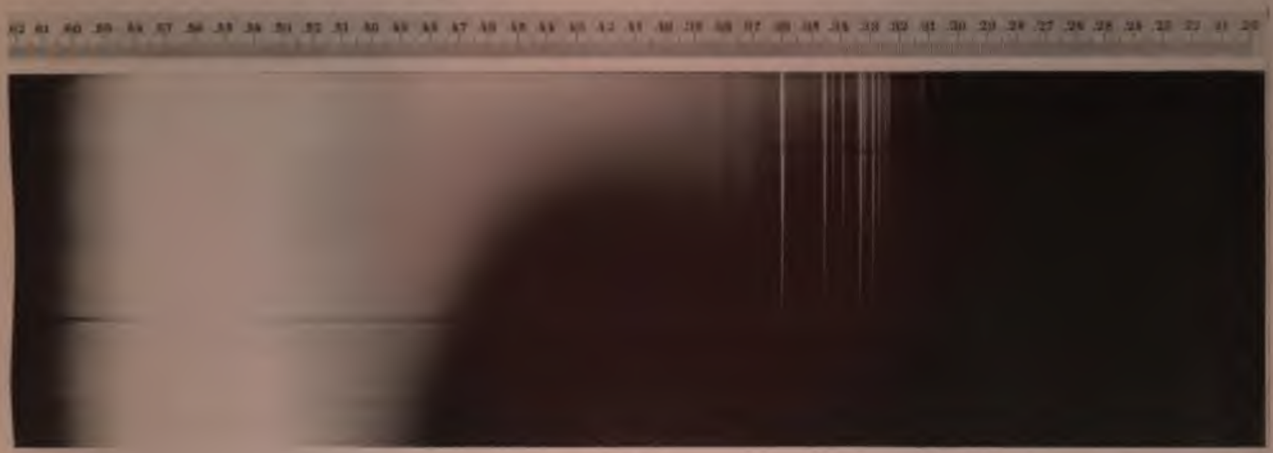


FIG. 14.

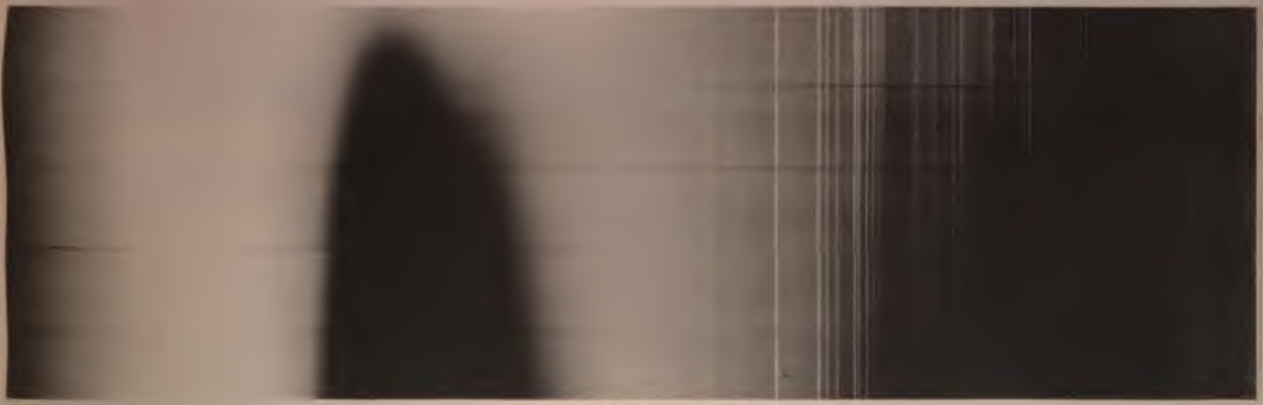


FIG. 15.

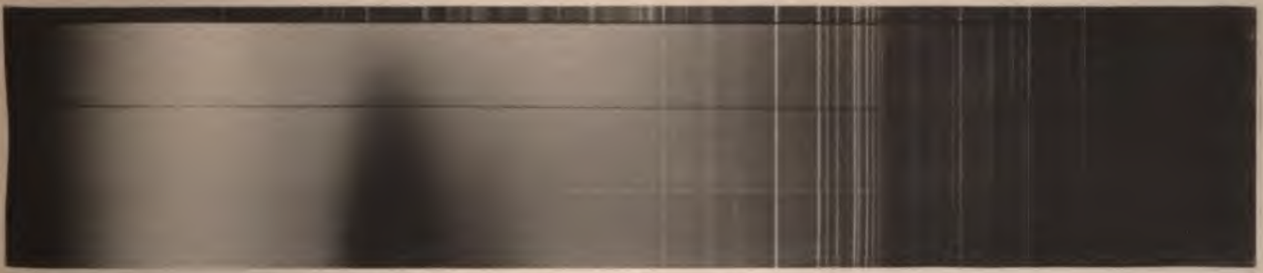


FIG. 16.

FIG. 14. SEE NO. 119. ALIZARINE RED S.
FIG. 15. SEE NO. 107. URANINE.
FIG. 16. SEE NO. 107. URANINE.

HELIOTYPE CO., BOSTON.

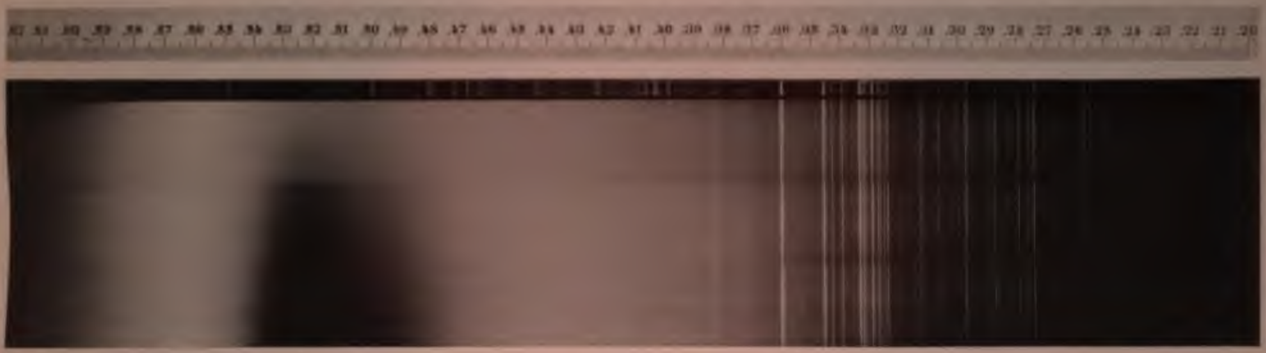


FIG. 17.

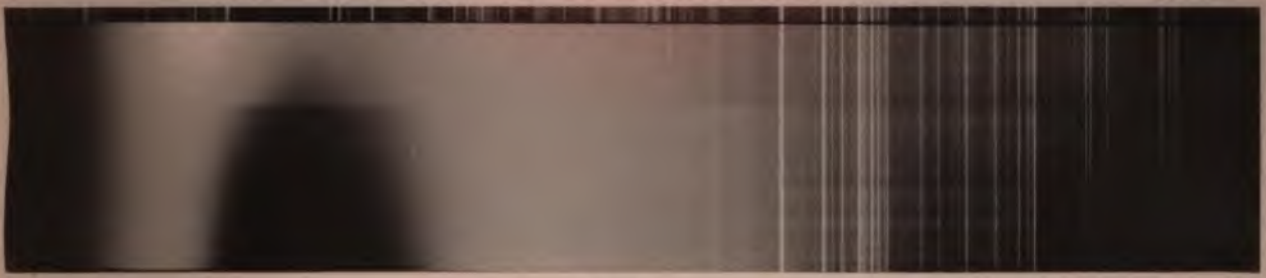


FIG. 18.

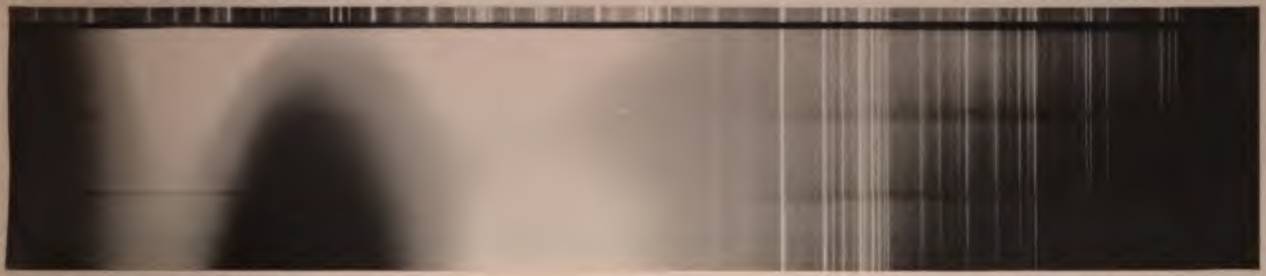


FIG. 19.

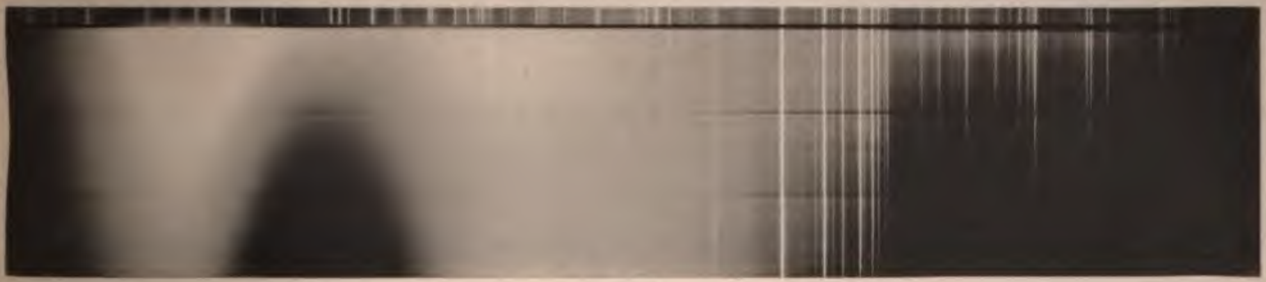


FIG. 20.

FIG. 17. SEE NO. 109. EOSINE À L'ALCOOL.
FIG. 18. SEE NO. 113. CYANOSINE.
FIG. 19. SEE NO. 106. ACID ROSAMINE A.
FIG. 20. SEE NO. 127. NAPHTHALENE RED.

HELIOTYPE CO., BOSTON.

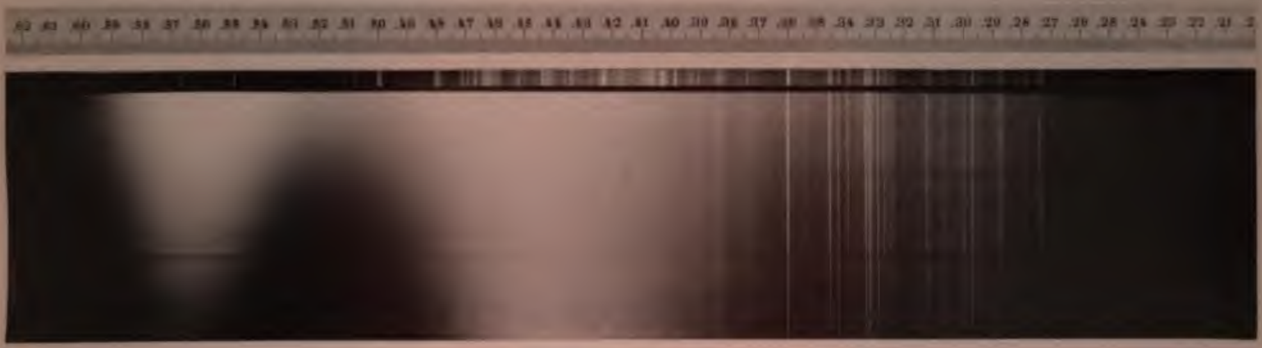


FIG. 21.

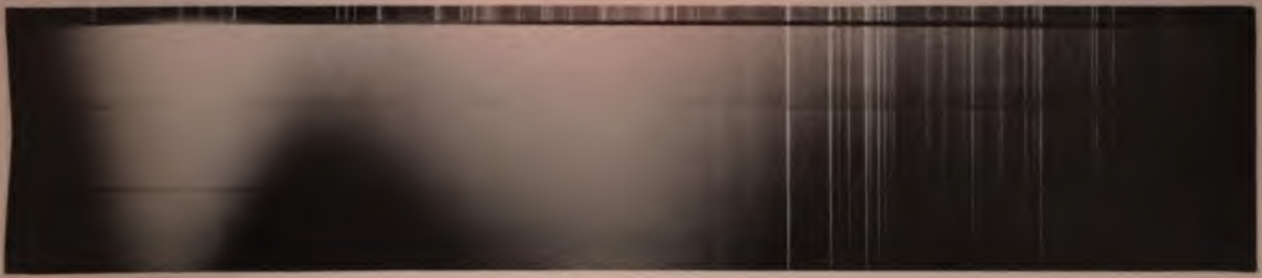


FIG. 22.

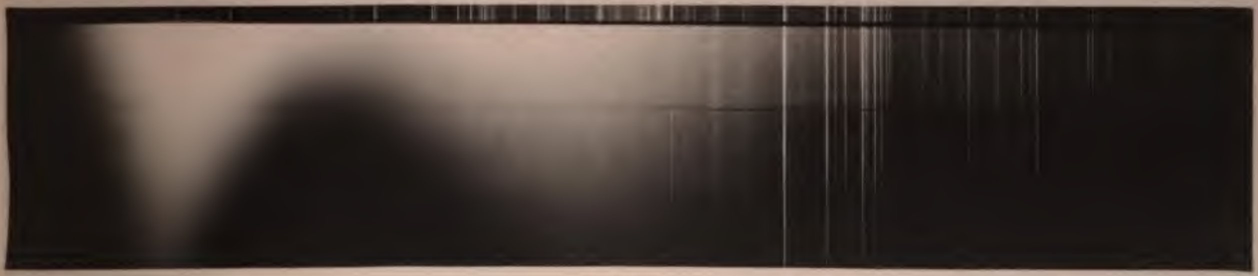


FIG. 23.

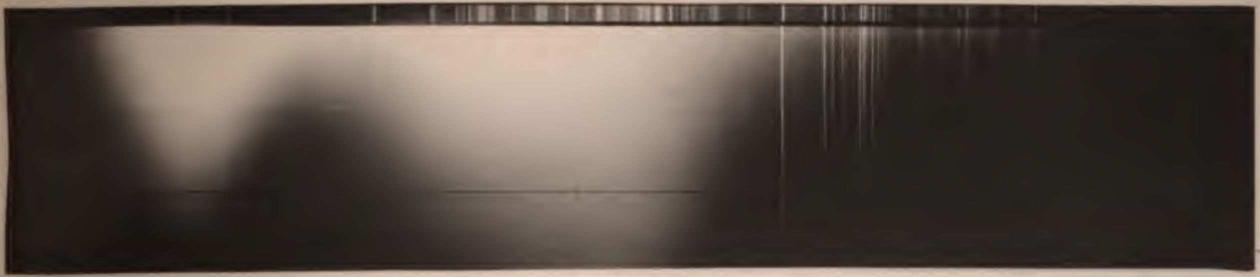


FIG. 24.

FIG. 21. SEE NO. 46. CLOTH RED O.
FIG. 22. SEE NO. 74. BENZOPURPURINE 10 B.
FIG. 23. SEE NO. 60. CONGO CORINTH G.
FIG. 24. SEE NO. 134. ALIZARINE ORANGE.

HELIOTYPE CO., BOSTON.

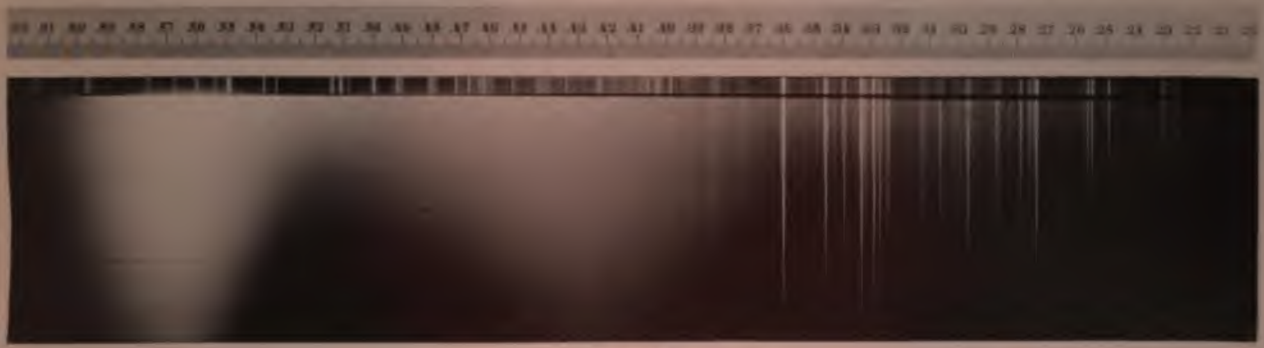


FIG. 25.

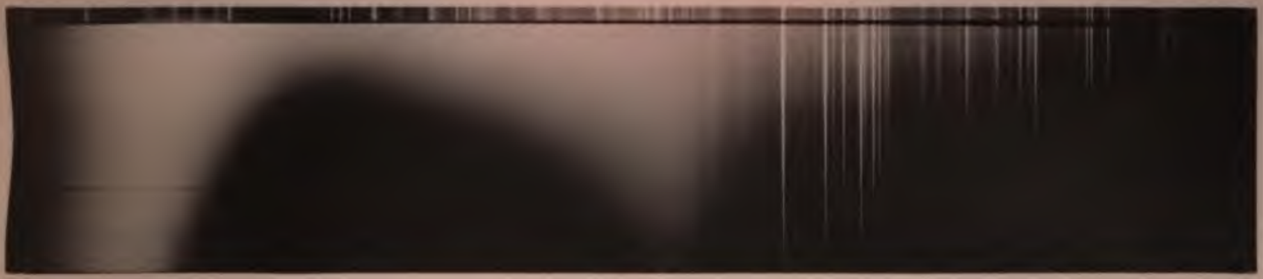


FIG. 26.

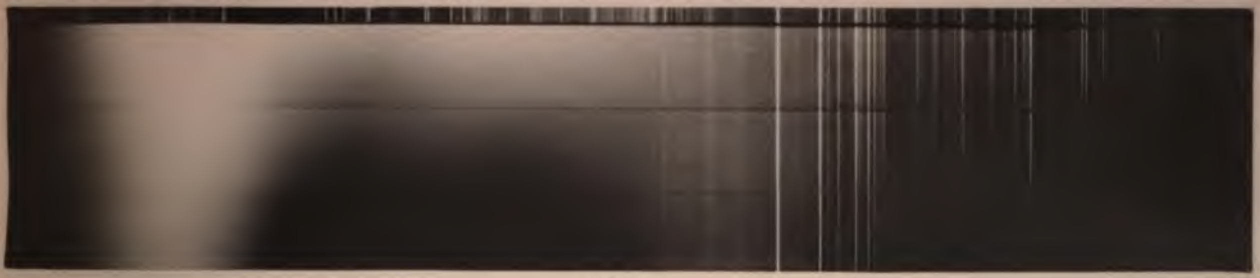


FIG. 27.

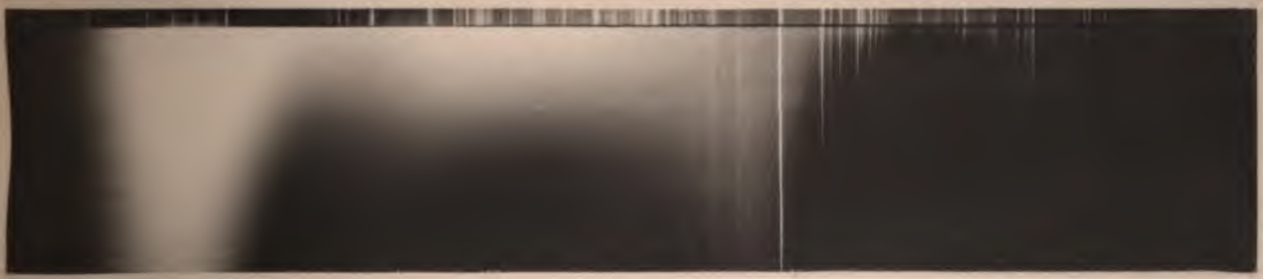


FIG. 28.

FIG. 25. SEE NO. 45. CLOTH RED G.
FIG. 26. SEE NO. 69. BRILLIANT PURPURINE R.
FIG. 27. SEE NO. 34. FAST RED A.
FIG. 28. SEE NO. 120. ALIZARINE BLUE S.

HELIOTYPE CO., BOSTON.

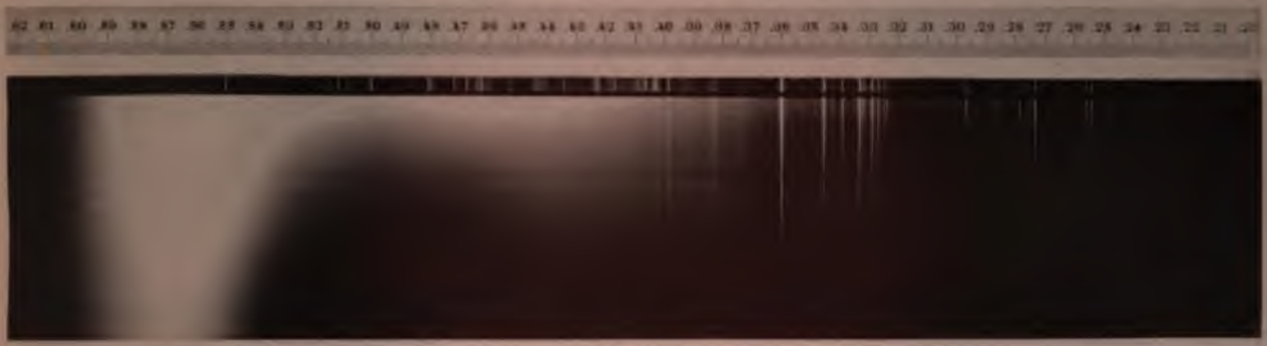


FIG. 29.

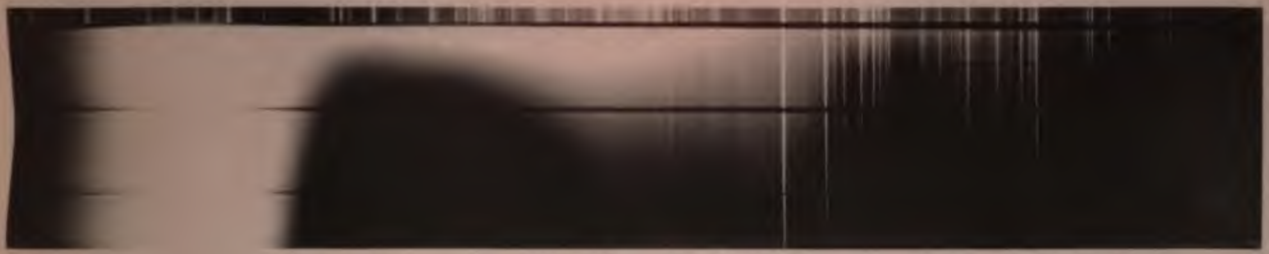


FIG. 30.

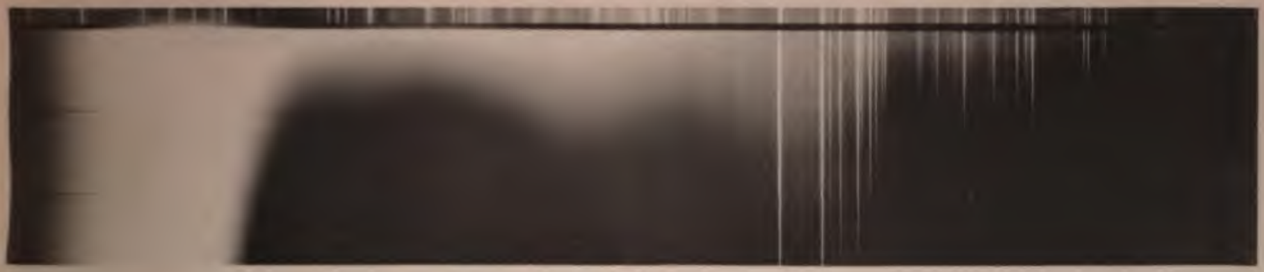


FIG. 31.

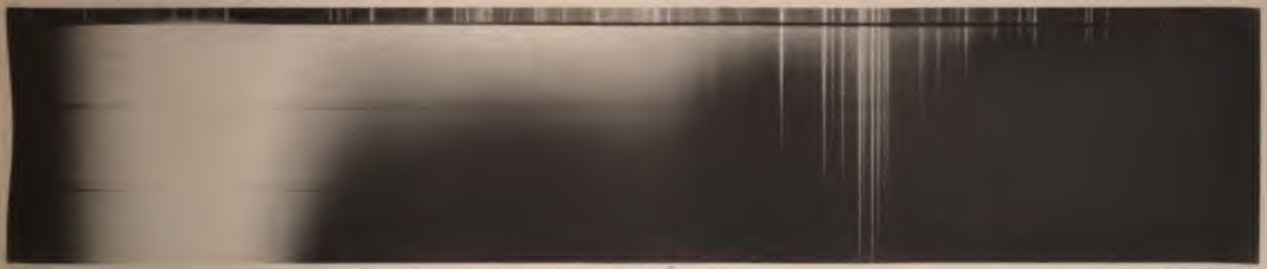


FIG. 32.

FIG. 29. SEE NO. 23. EMIN RED.
FIG. 30. SEE NO. 10. ORANGE G.
FIG. 31. SEE NO. 15. BRILLIANT ORANGE G.
FIG. 32. SEE NO. 117. PHOSPHINE.

HELIOTYPE CO., BOSTON.

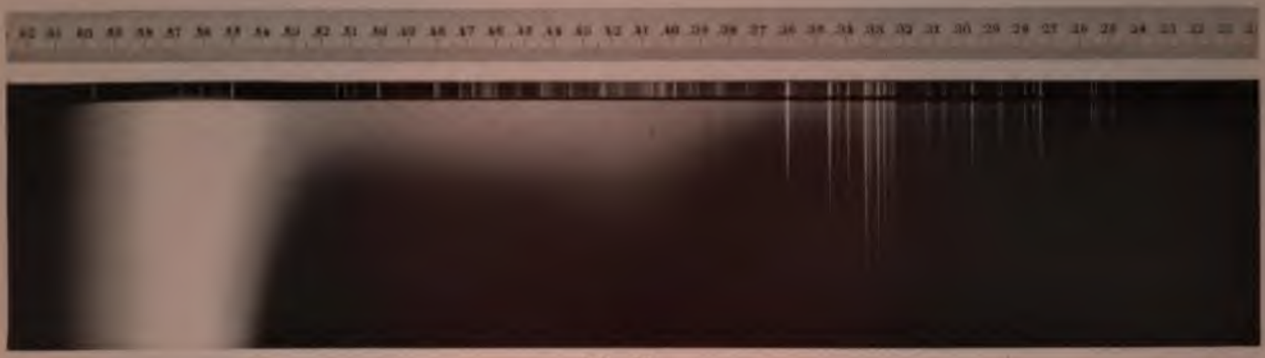


FIG. 33.

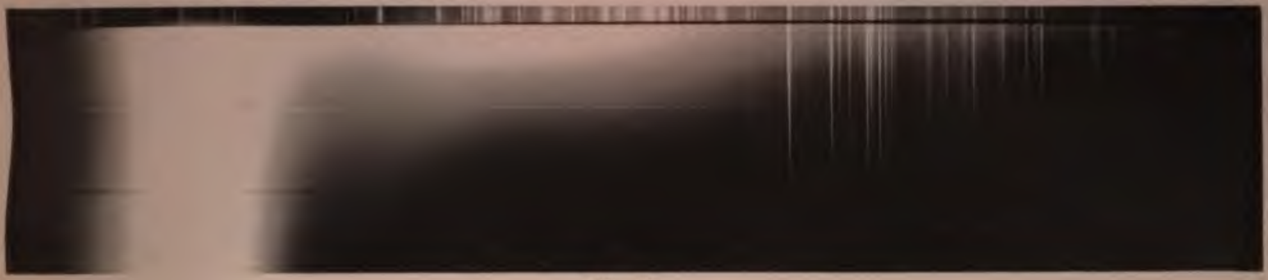


FIG. 34.

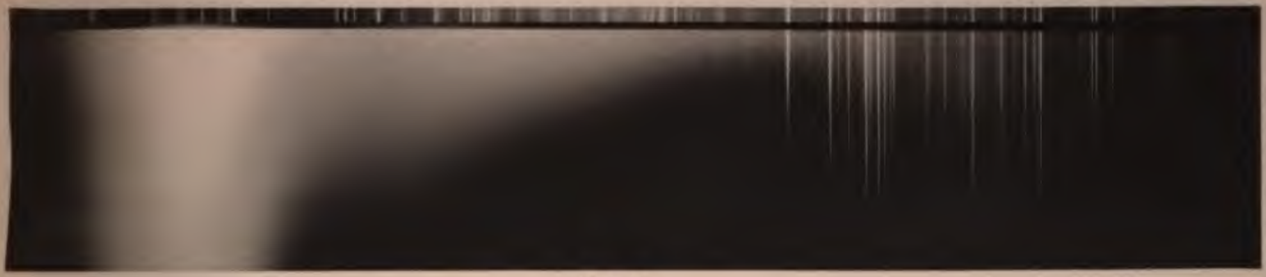


FIG. 35.

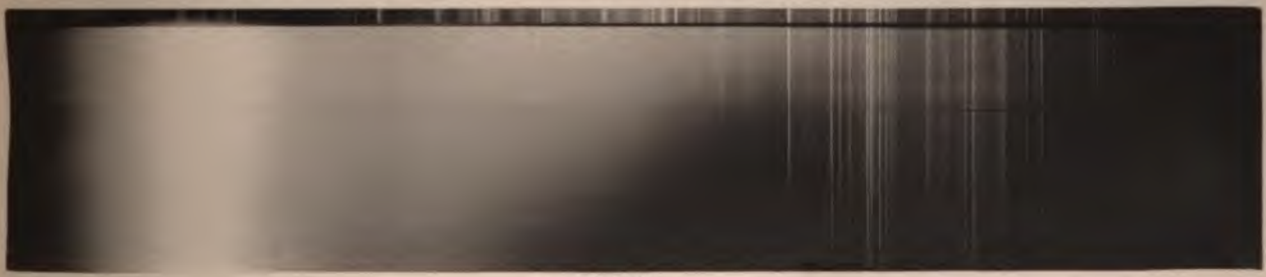


FIG. 36.

FIG. 33. SEE NO. 56. CONGO ORANGE G.
FIG. 34. SEE NO. 141. DIANIL ORANGE G.
FIG. 35. SEE NO. 77. CONGO BROWN G.
FIG. 36. SEE NO. 77. CONGO BROWN G.

HELIOTYPE CO., BOSTON.

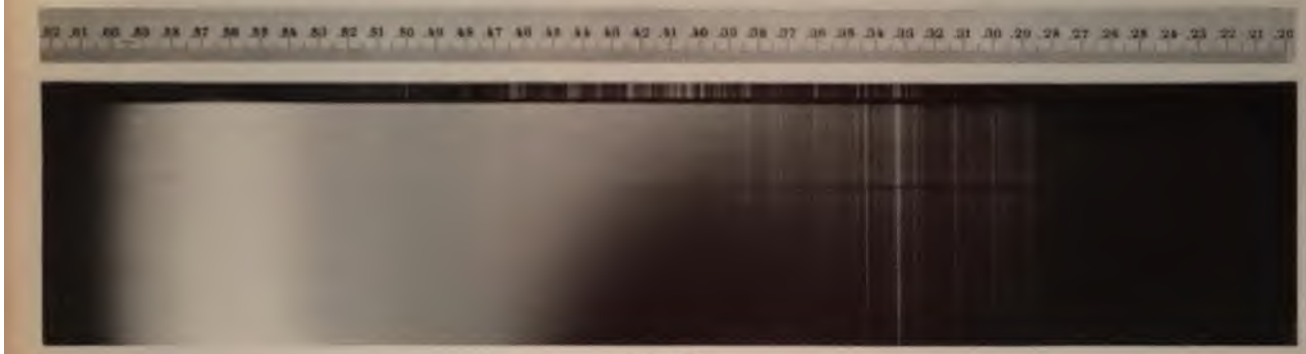


FIG. 37.

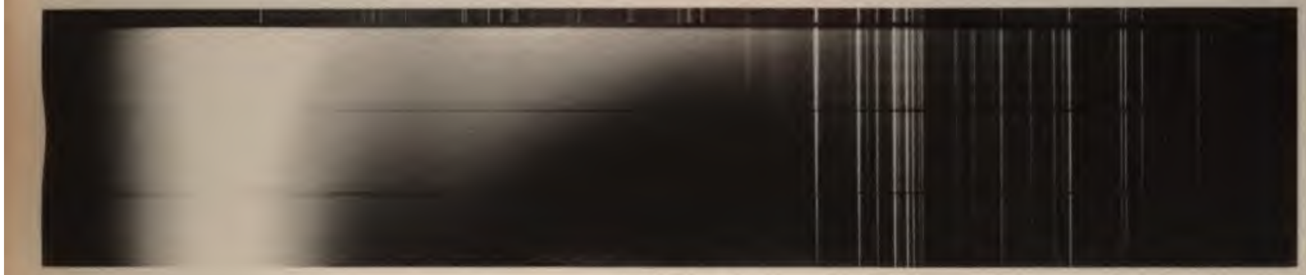


FIG. 38.

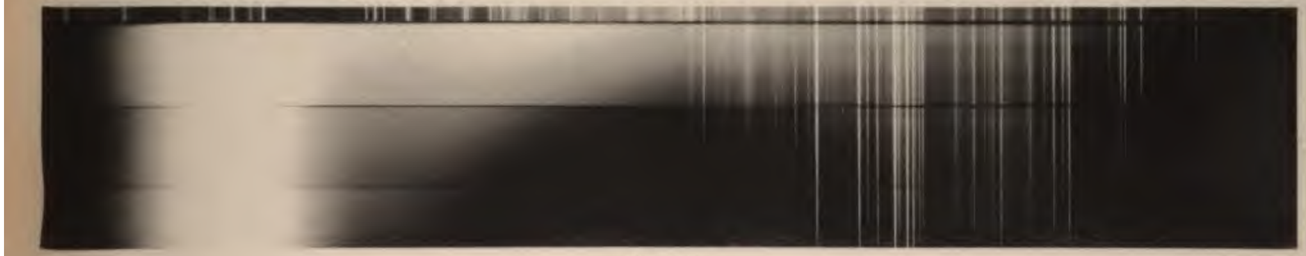


FIG. 39.

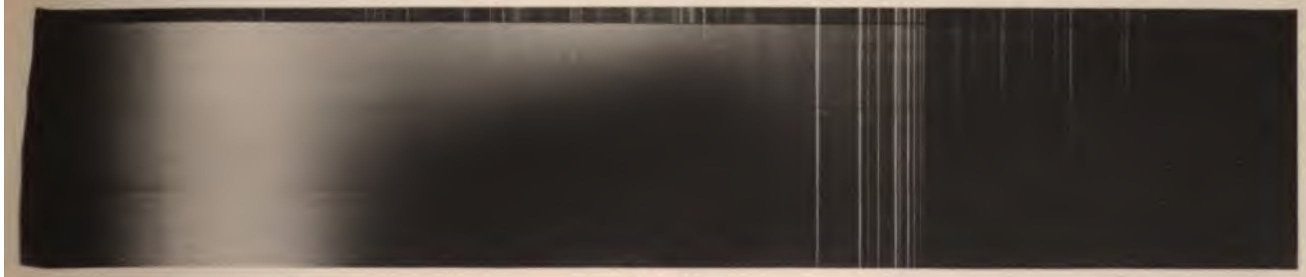


FIG. 40.

FIG. 37.	SEE NO. 81.	CURCUMINE S.
FIG. 38.	SEE NO. 41.	RESORCINE BROWN.
FIG. 39.	SEE NO. 8.	AURANTIA.
FIG. 40.	SEE NO. 32.	METANIL YELLOW.

HELIOTYPE CO., BOSTON.

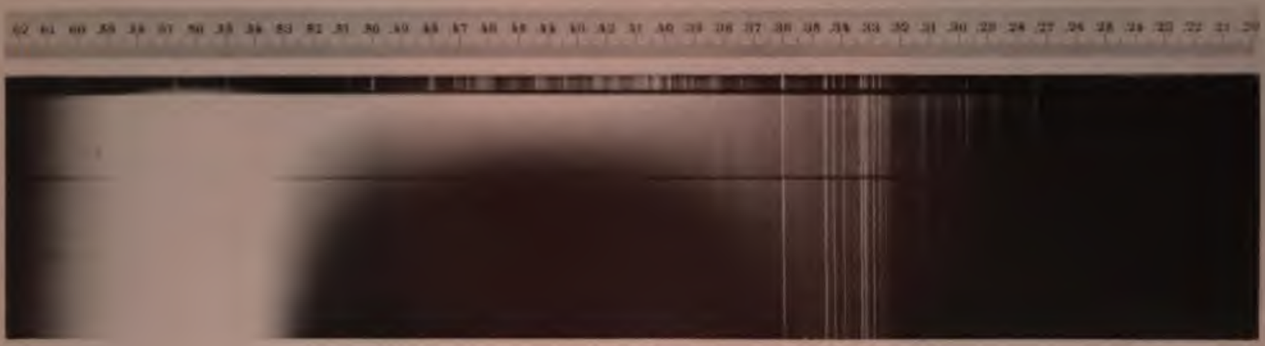


FIG. 41.

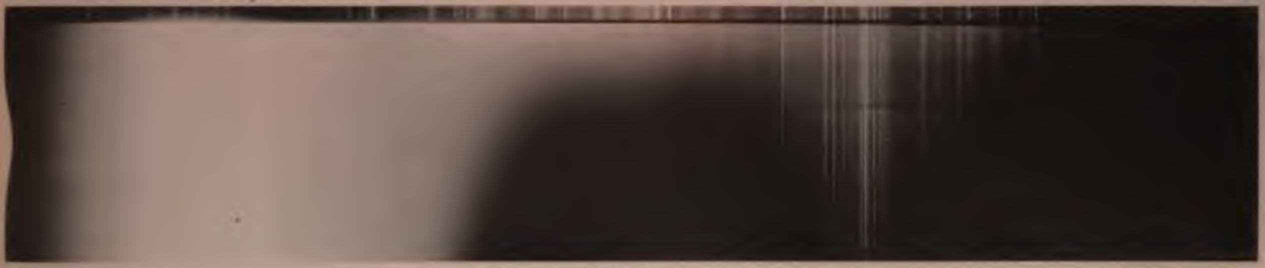


FIG. 42.

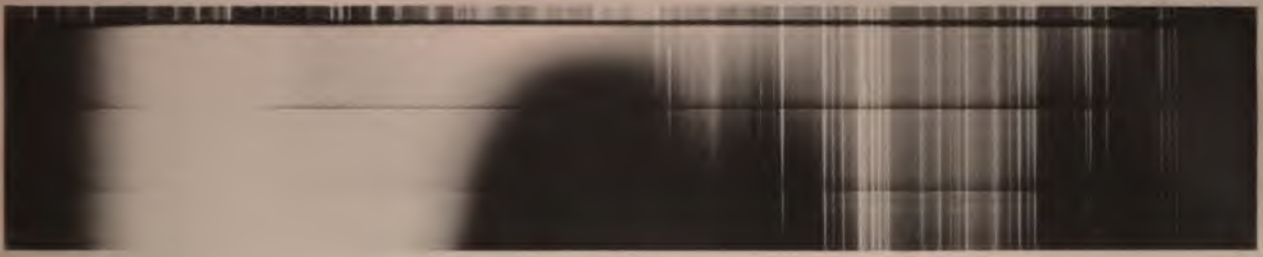


FIG. 43.

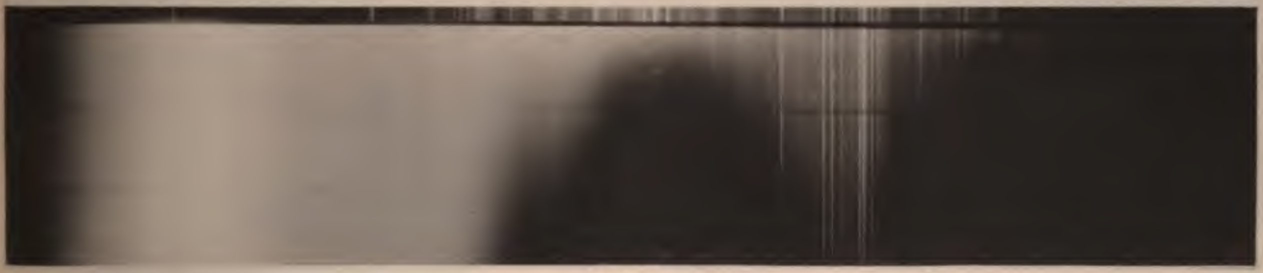


FIG. 44.

FIG. 41. SEE NO. 28. METHYL ORANGE III
FIG. 42. SEE NO. 7. NAPHTHOL YELLOW.
FIG. 43. SEE NO. 82. AURAMINE O.
FIG. 44. SEE NO. 131. QUINOLINE YELLOW O.

HELIOTYPE CO., BOSTON.

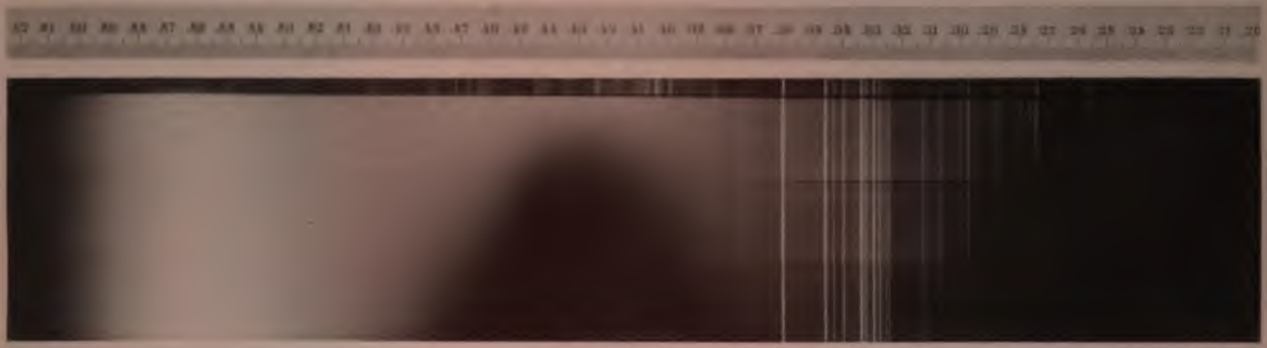


FIG. 45.

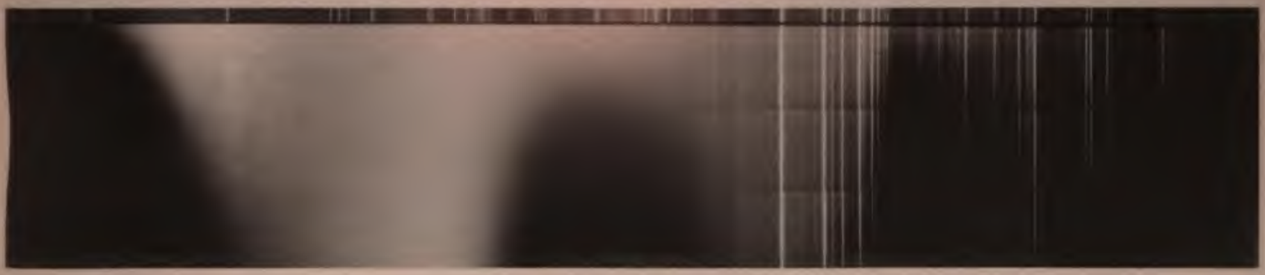


FIG. 46.

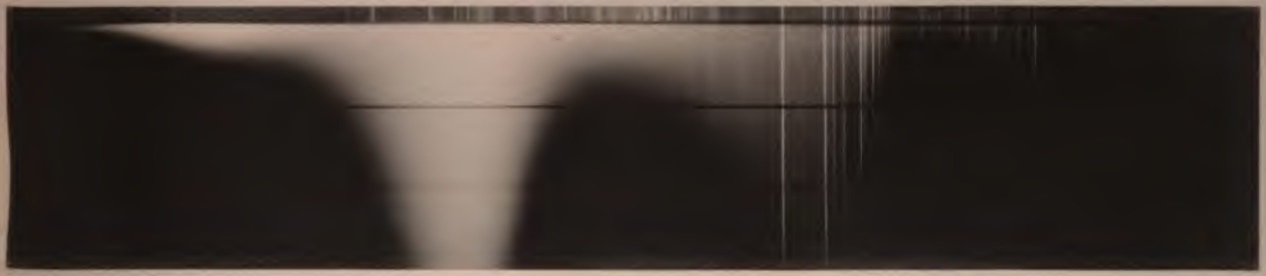


FIG. 47.

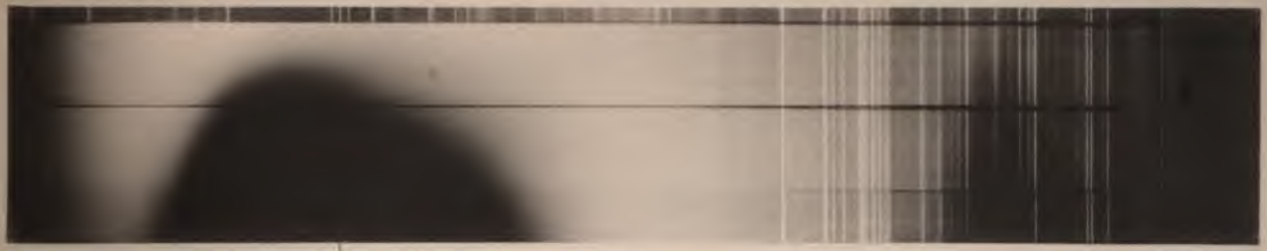


FIG. 48.

FIG. 45. SEE NO. 136. AUROPHOSPHINE 4 G.
FIG. 46. SEE NO. 86. ACID GREEN, CONC'.
FIG. 47. SEE NO. 94. METHYL GREEN.
FIG. 48. SEE NO. 87. FUCHSINE.

HELIOTYPE CO., BOSTON.

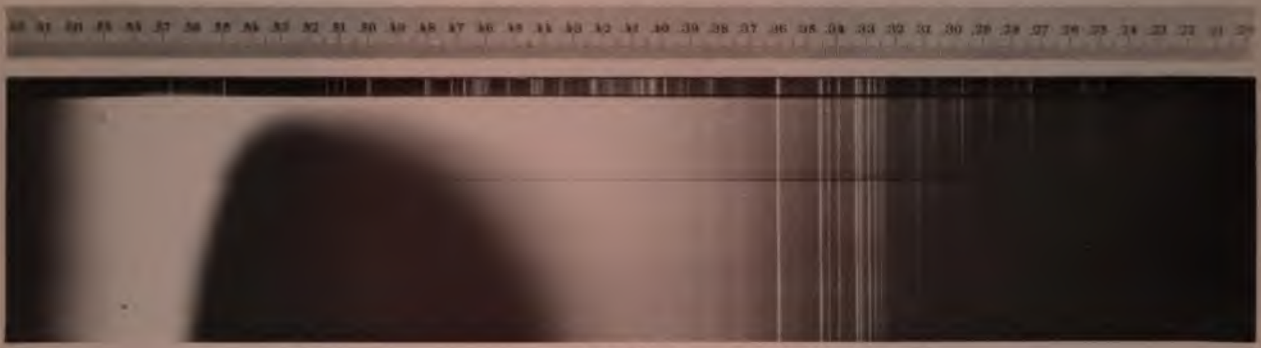


FIG. 49.

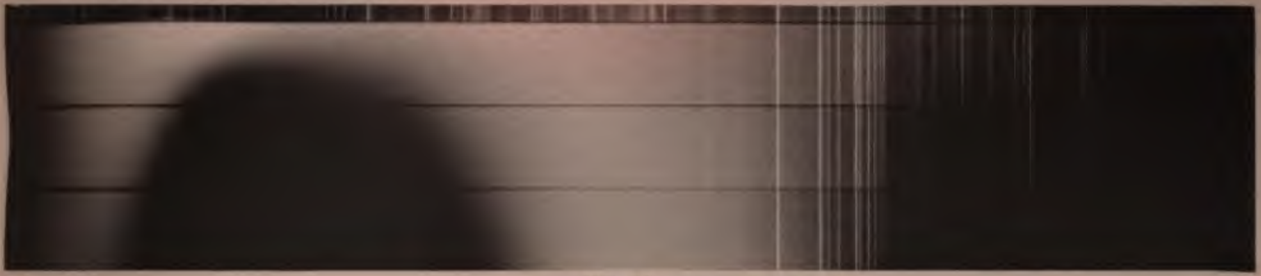


FIG. 50.

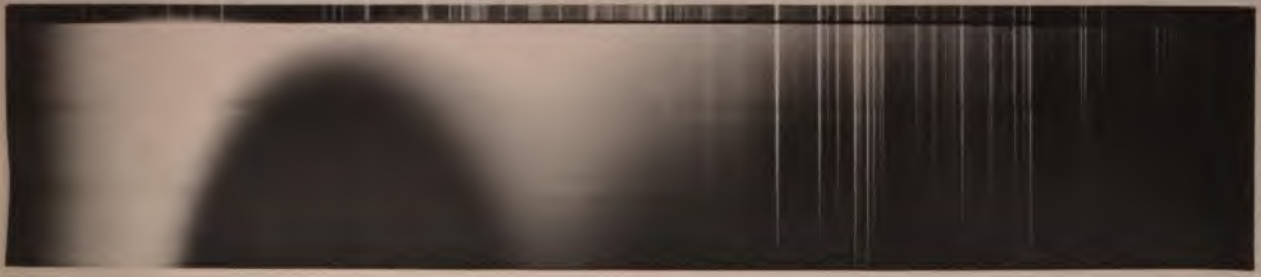


FIG. 51.

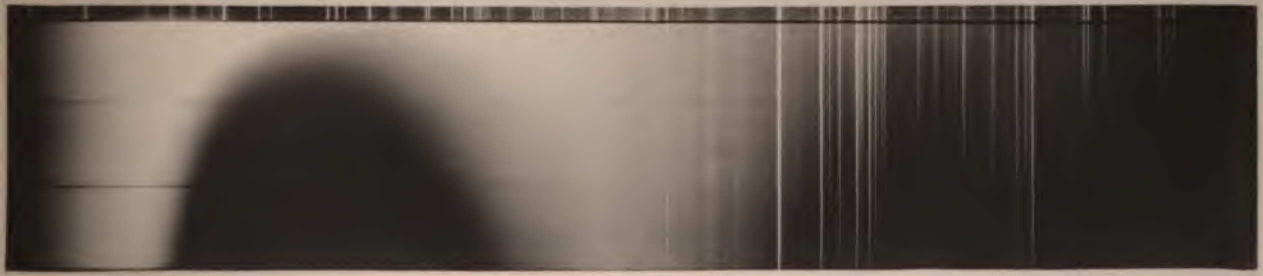


FIG. 52.

FIG. 49. SEE NO. 100. CORALLINE RED.
FIG. 50. SEE NO. 88. NEW MAGENTA.
FIG. 51. SEE NO. 48. PONCEAU 4 R B.
FIG. 52. SEE NO. 21. EOSAMINE B.

HELIOTYPE CO., BOSTON.

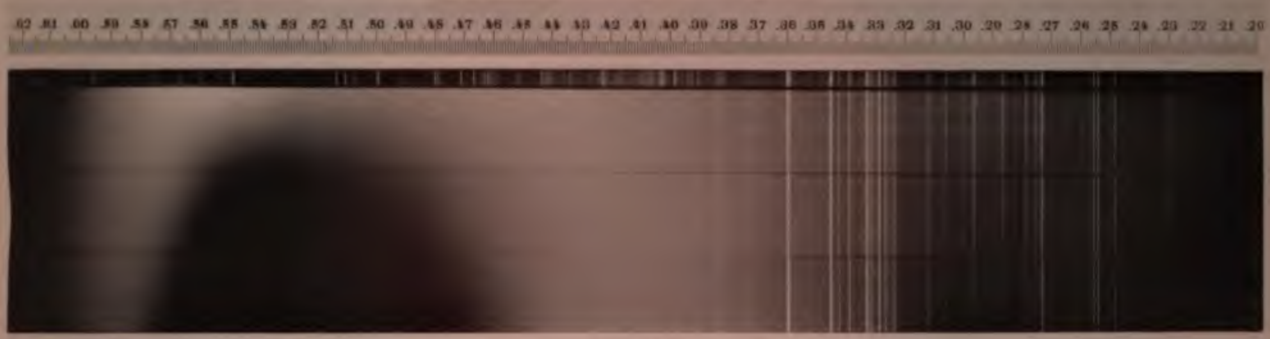


FIG. 53.

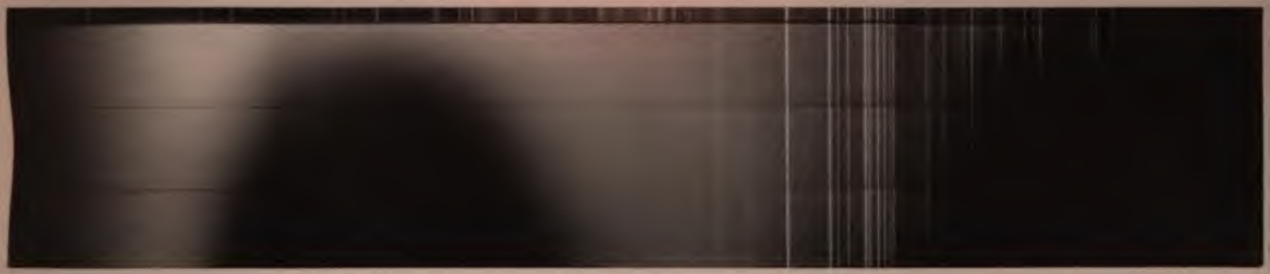


FIG. 54.

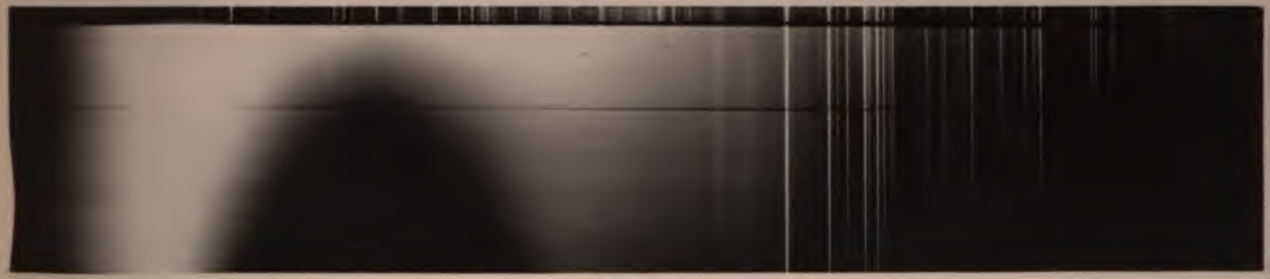


FIG. 55.

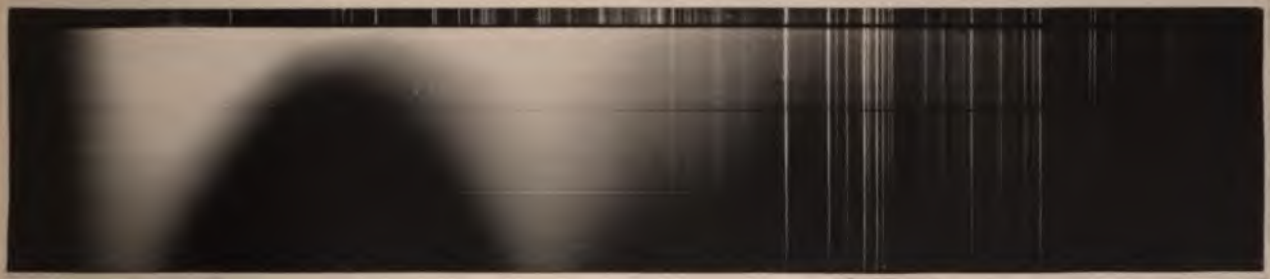


FIG. 56.

FIG. 53. SEE NO. 95. FUCHSINE S.
FIG. 54. SEE NO. 122. PHENOSAFRANINE.
FIG. 55. SEE NO. 17. PONCEAU 3 R.
FIG. 56. SEE NO. 51. PONCEAU 6 R B.

HELIOTYPE CO., BOSTON.

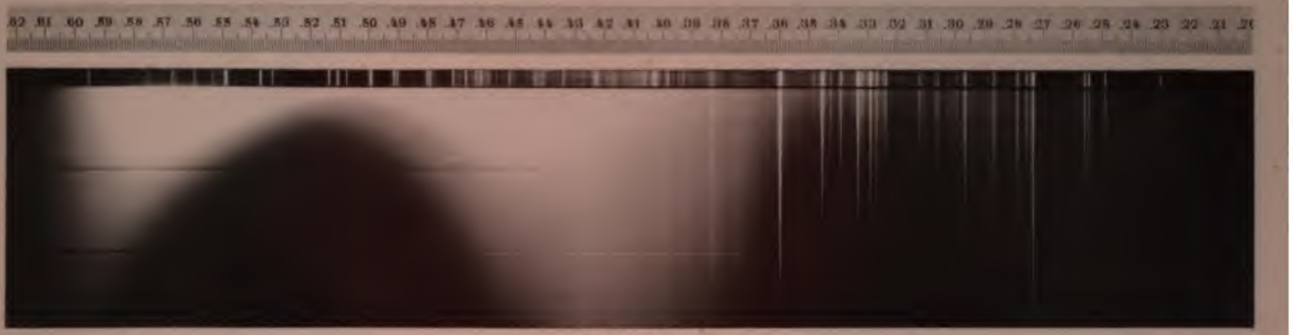


FIG. 57.

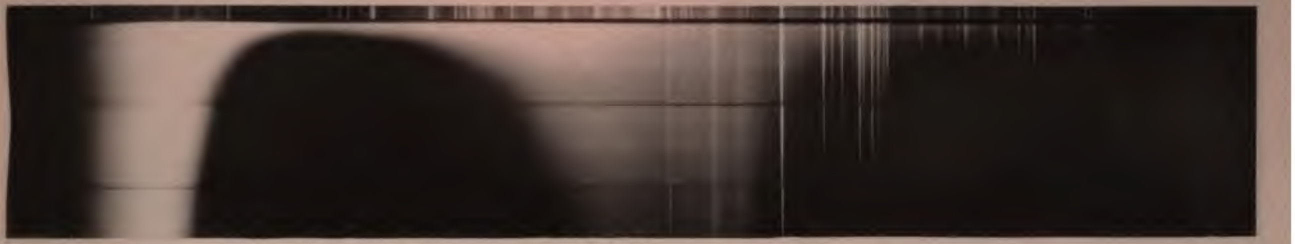


FIG. 58.

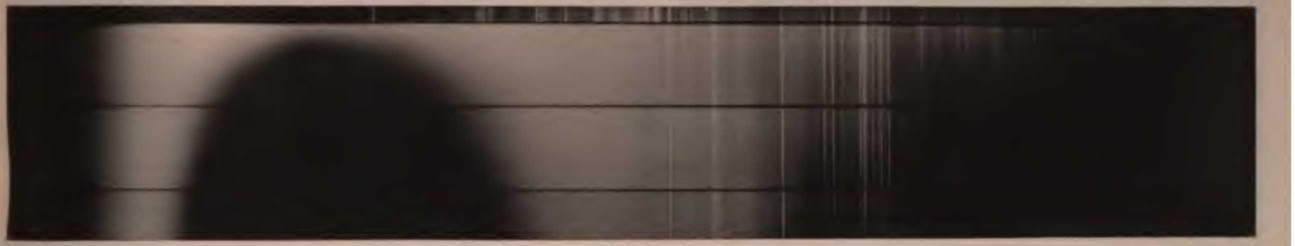


FIG. 59.

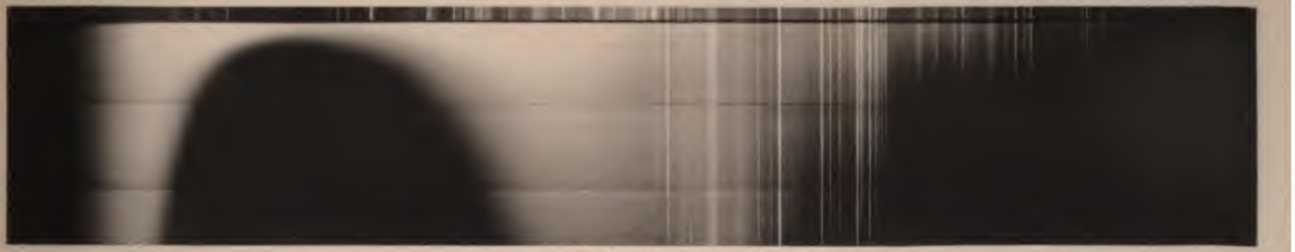


FIG. 60.

FIG. 57. SEE NO. 22. ERIKA B.
FIG. 58. SEE NO. 108. EOSINE, YELLOWISH.
FIG. 59. SEE NO. 112. ERYTHROSINE.
FIG. 60. SEE NO. 115. PHLOXINE.

HELIOTYPE CO., BOSTON.

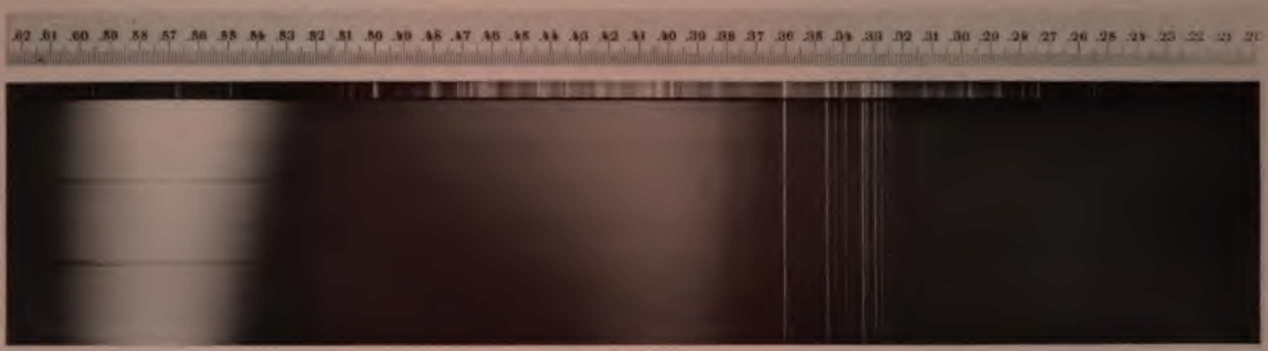


FIG. 61.

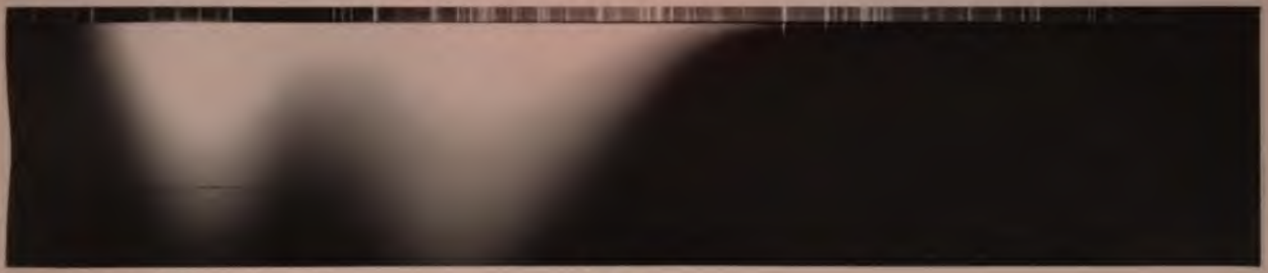


FIG. 62.

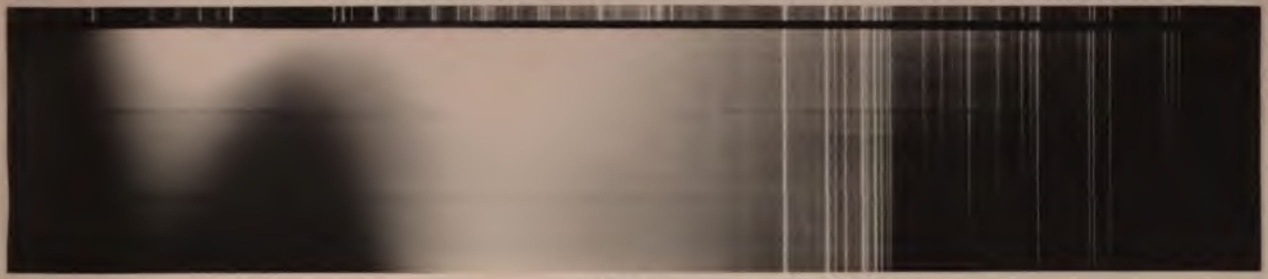


FIG. 63.

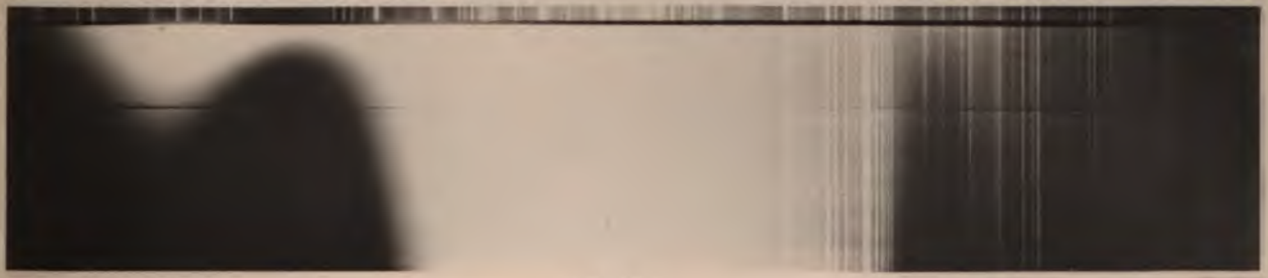
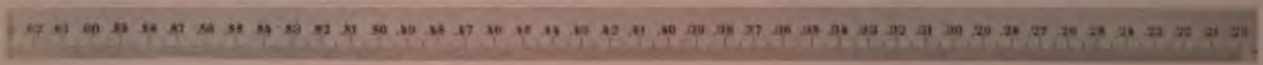


FIG. 64.

FIG. 61. SEE NO. 68. DIAMINE RED 3 B.
FIG. 62. SEE NO. 96. RED VIOLET 5 R S.
FIG. 63. SEE NO. 104. FAST ACID VIOLET B.
FIG. 64. SEE NO. 91. ETHYL VIOLET.

HELIOTYPE CO., BOSTON.



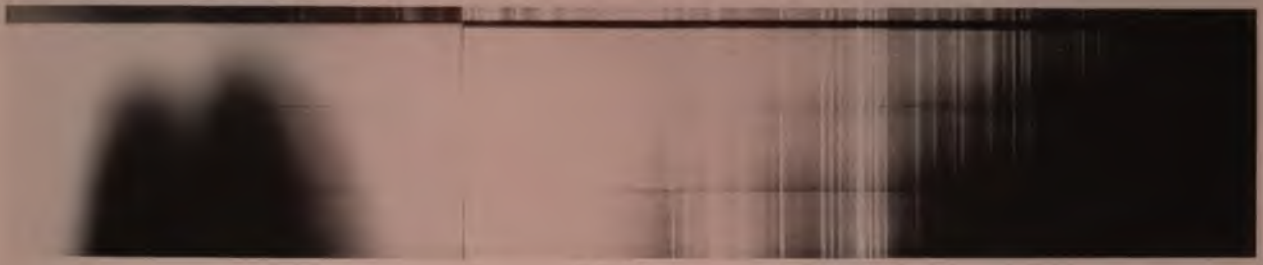


FIG. 65.

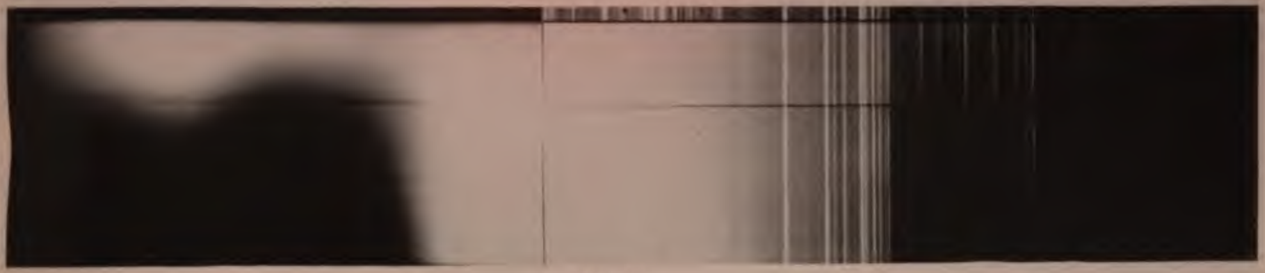


FIG. 66.

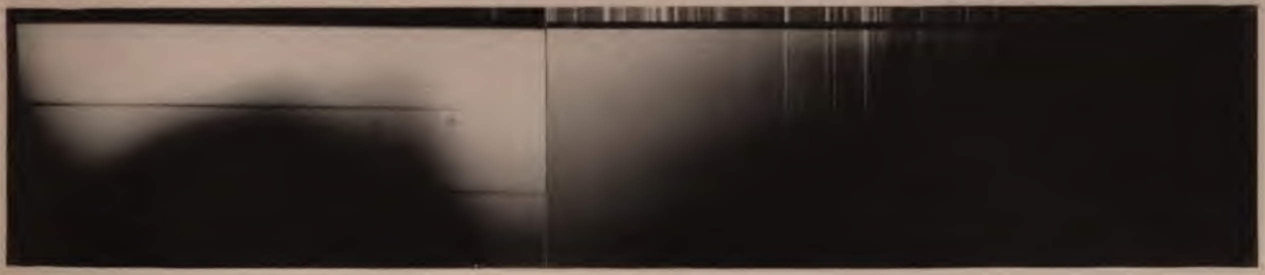


FIG. 67.

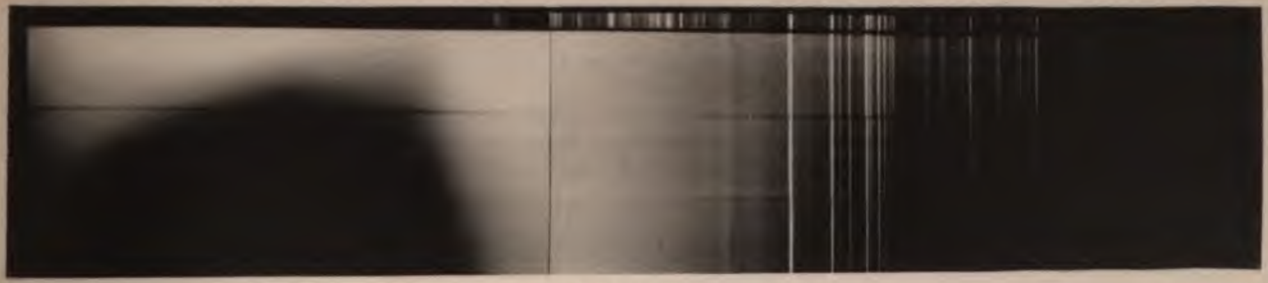


FIG. 68.

FIG. 65. SEE NO. 103. RHODAMINE B.
FIG. 66. SEE NO. 92. METHYL VIOLET 6B.
FIG. 67. SEE NO. 50. WOOL BLACK.
FIG. 68. SEE NO. 124. HELIOTROPE 2B.

HELIOTYPE CO., BOSTON.

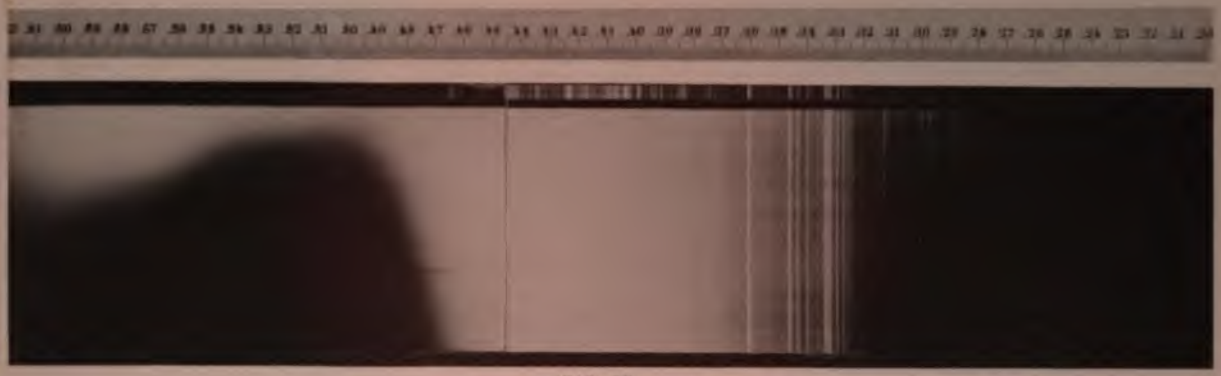


FIG. 69.

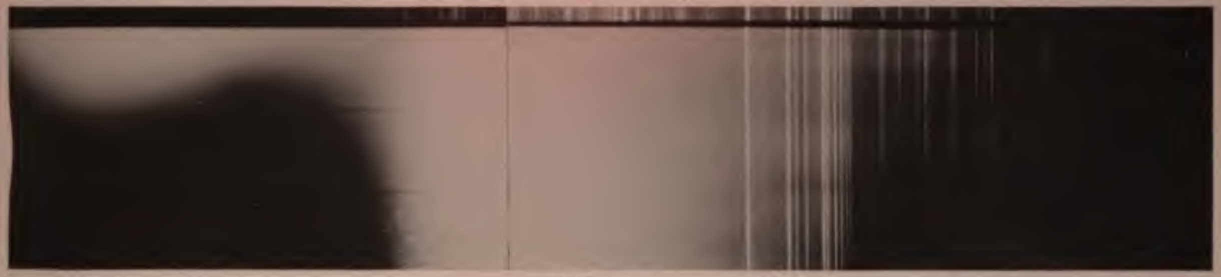


FIG. 70.

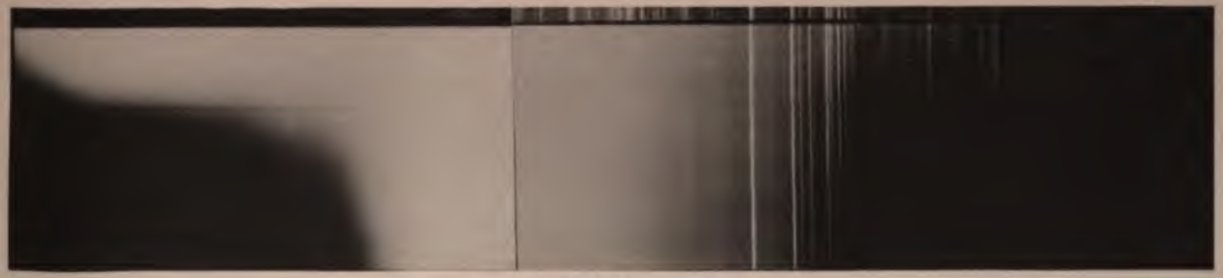


FIG. 71.

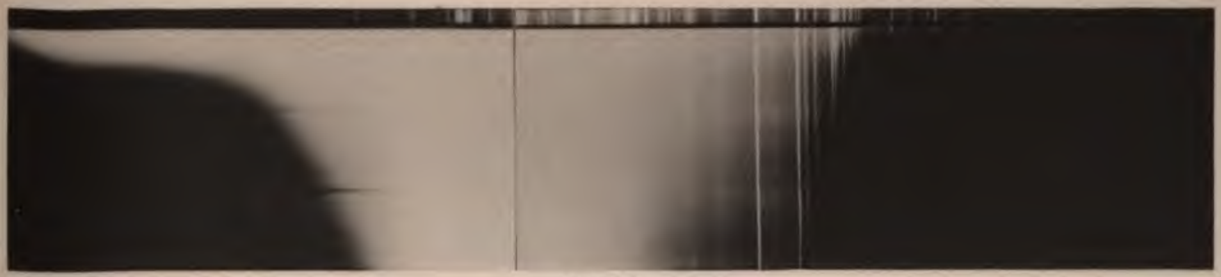


FIG. 72.

FIG. 69. SEE NO. 89. DAHLIA.
FIG. 70. SEE NO. 102. VICTORIA BLUE 4R.
FIG. 71. SEE NO. 99. CHINA BLUE.
FIG. 72. SEE NO. 97. ALKALI BLUE 6B.

HELIOTYPE CO., BOSTON.

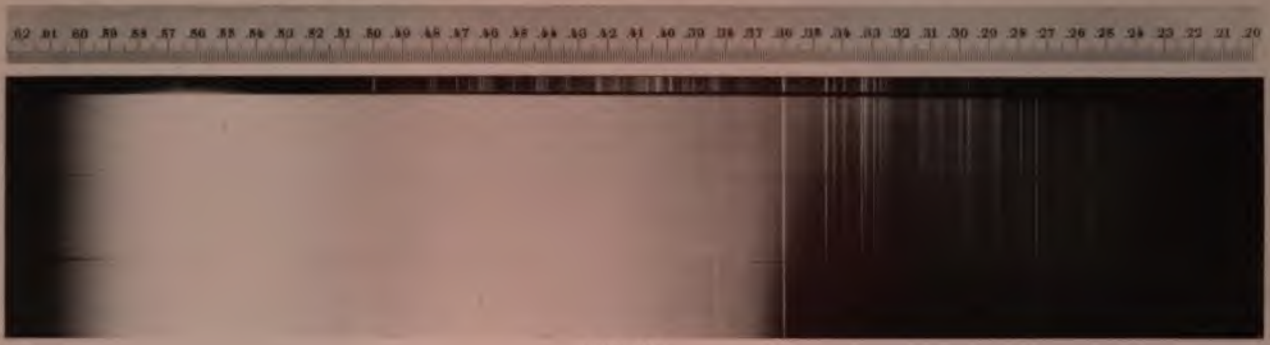


FIG. 73.

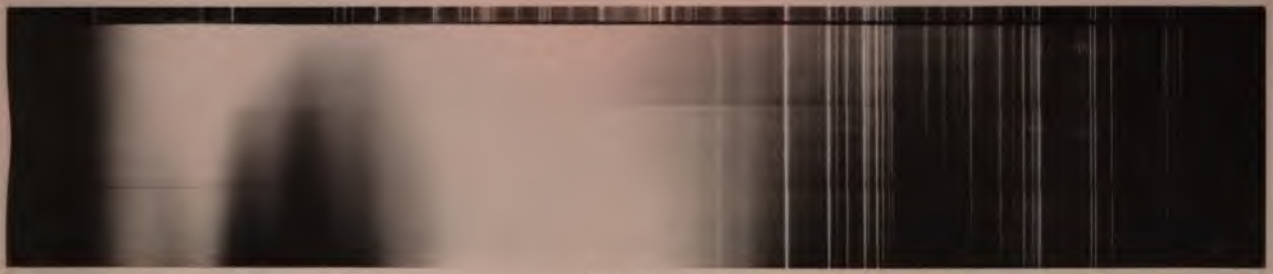


FIG. 74.

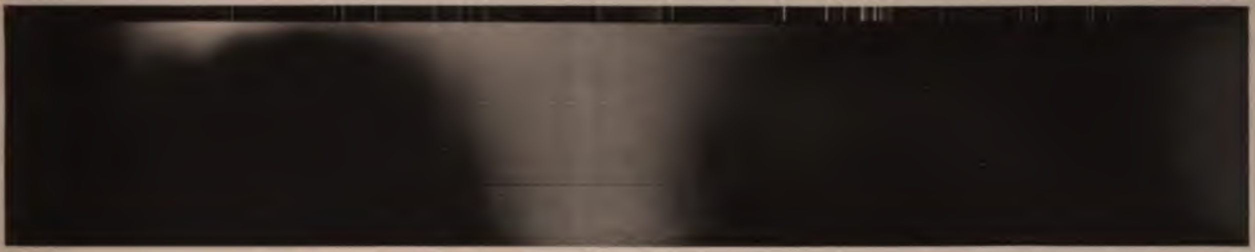


FIG. 75.

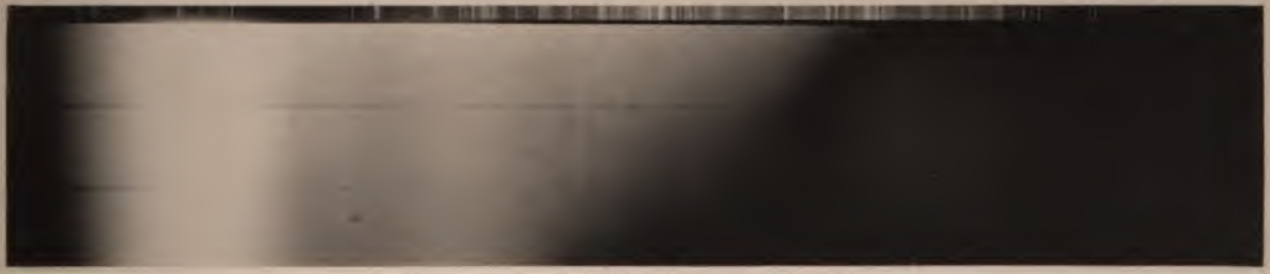


FIG. 76.

FIG. 73. SEE NO. 149. AESCULINE.
FIG. 74. SEE NO. 179. POTASSIUM PERMANGANATE.
FIG. 75. SEE NO. 179. POTASSIUM PERMANGANATE.
FIG. 76. SEE NO. 182. SODIUM NITROPRUSSÏD.

HELIOTYPE CO., BOSTON.

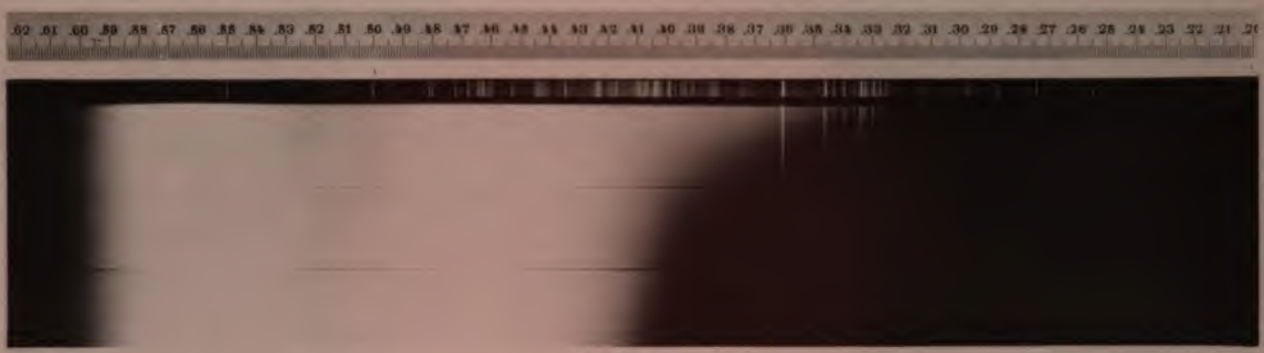


FIG. 77.

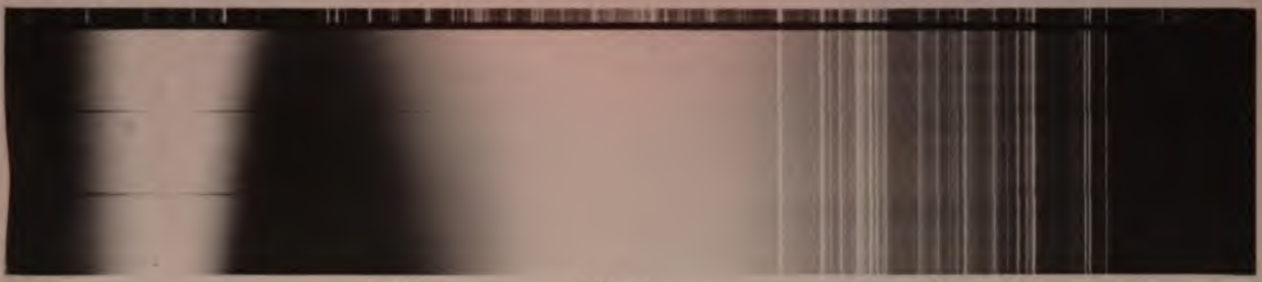


FIG. 78.

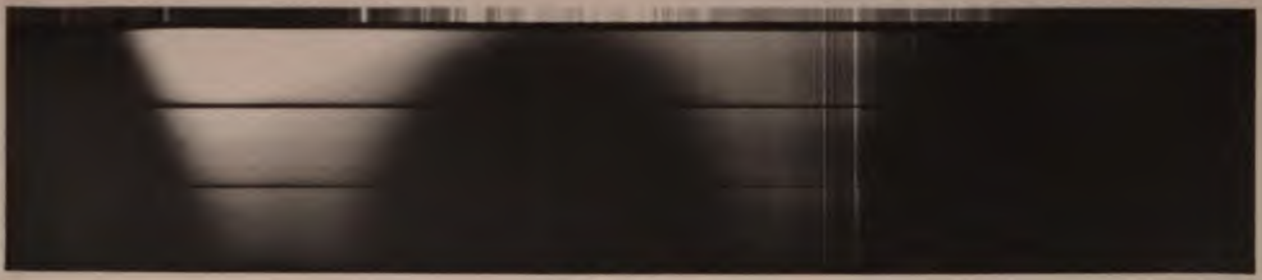


FIG. 79.

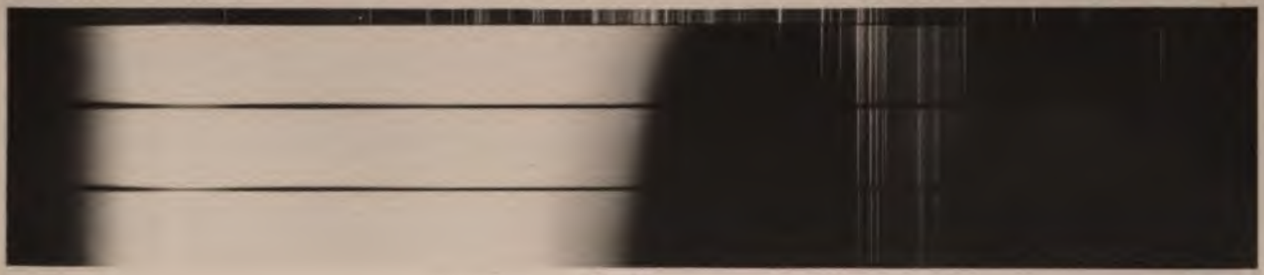


FIG. 80.

FIG. 77. SEE NO. 163. COPPER CHLORIDE.
FIG. 78. SEE NO. 156. COBALT CHLORIDE.
FIG. 79. SEE NO. 155. CHROMIUM CHLORIDE.
FIG. 80. SEE NO. 178. POTASSIUM CHROMATE.

HELIOTYPE CO., BOSTON.

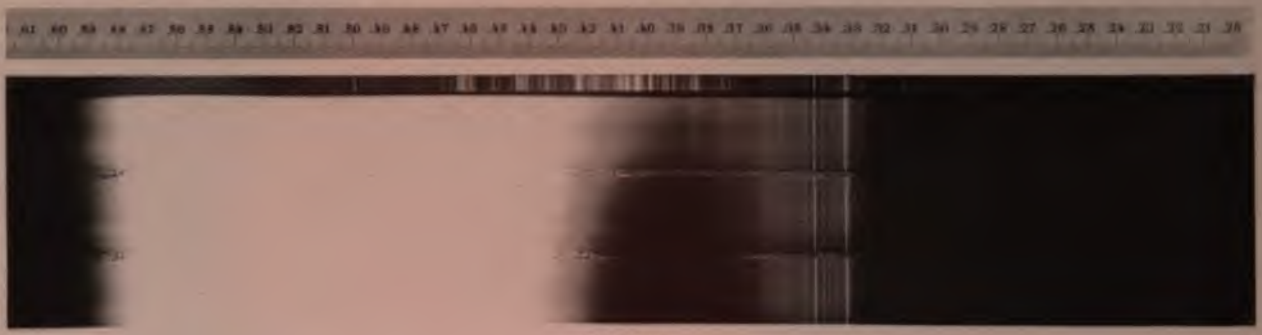


FIG. 81.

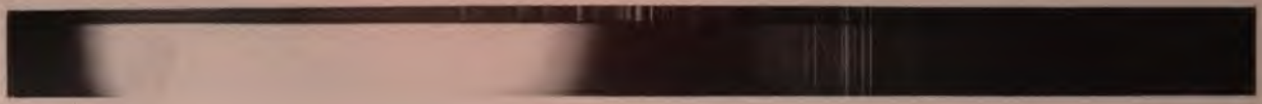


FIG. 82.

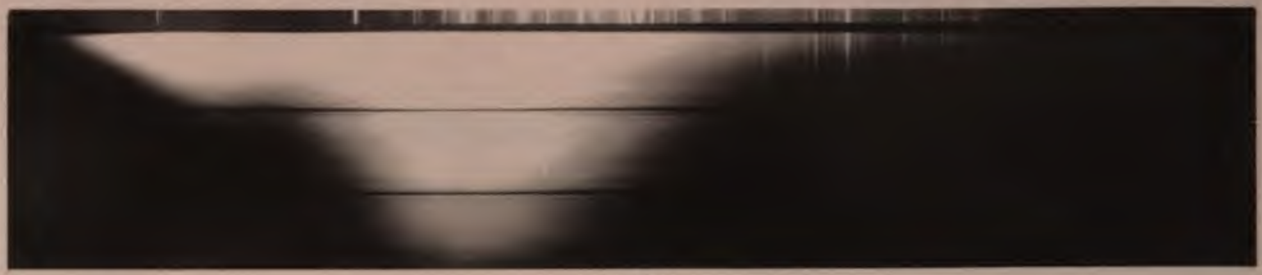


FIG. 83.

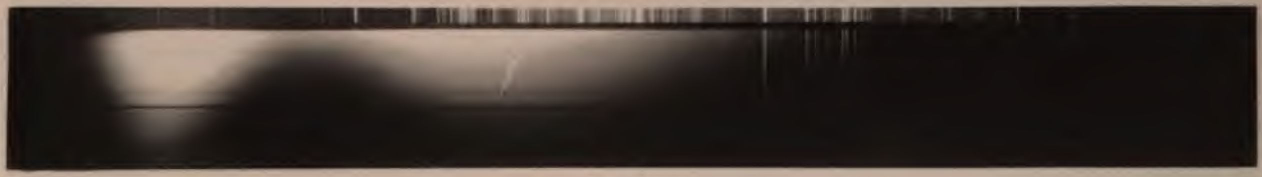


FIG. 84.

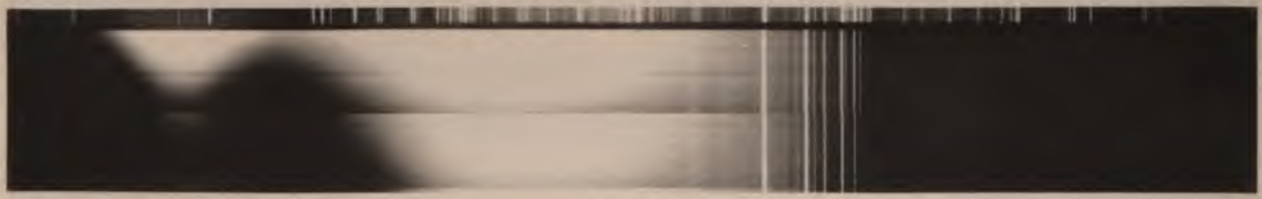


FIG. 85.

FIG. 81. SEE NO. 175. NICKEL NITRATE.
FIG. 82. SEE NO. 176. NICKEL SULPHATE.
FIG. 83. SEE NO. 172. LITMUS, BLUE.
FIG. 84. SEE NO. 172. LITMUS, RED.
FIG. 85. SEE NO. 161. COBALT GLASS.

HELIOTYPE CO., BOSTON.

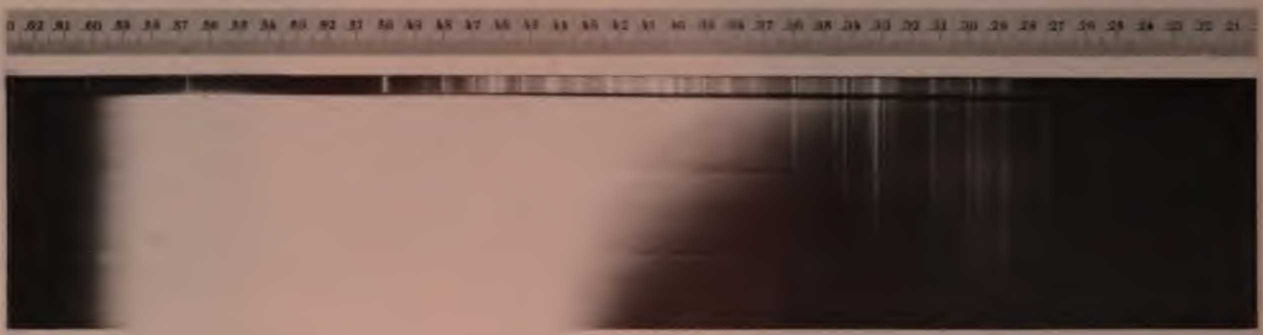


FIG. 86.

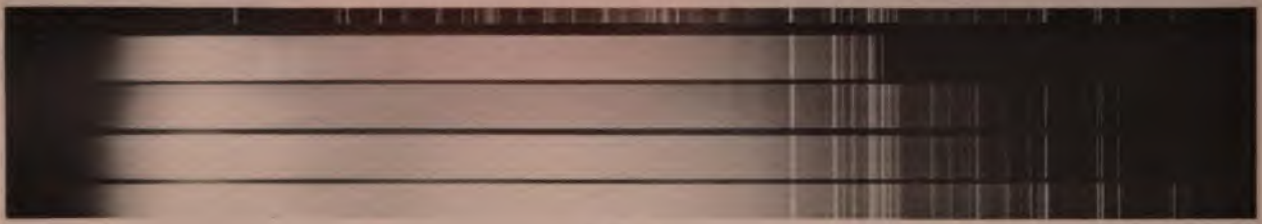


FIG. 87.

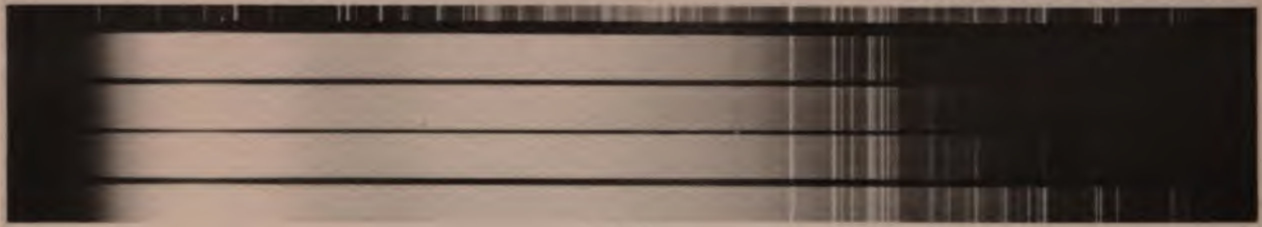


FIG. 88.

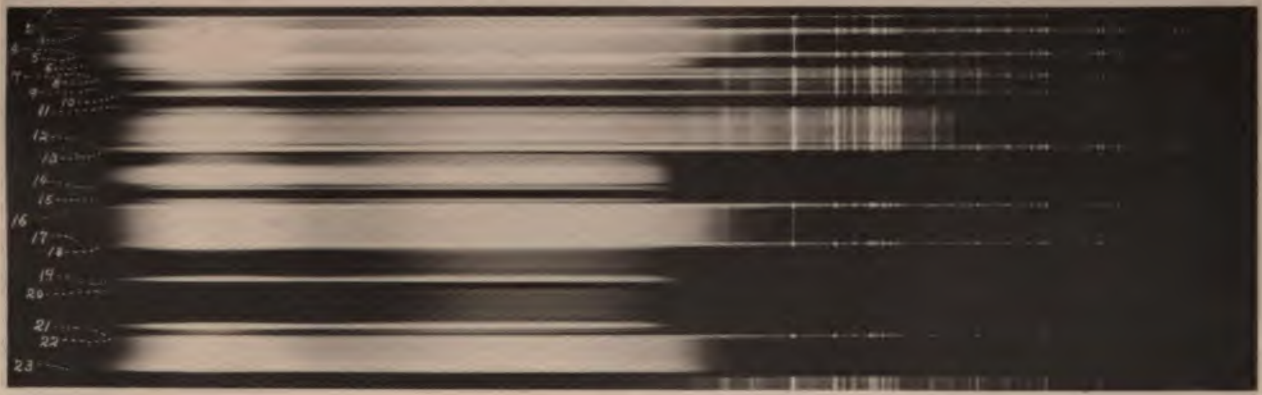


FIG. 89.

FIG. 86. SEE NO. 177. PICRIC ACID.
FIG. 87. SEE NO. 148. ACETONE, ETHYL ALCOHOL, ETC.
FIG. 88. SEE NO. 150. ALUMINIUM CHLORIDE, ETC.
FIG. 89. SEE NO. 154. CARBORUNDUM AND DIAMOND.

HELIOTYPE CO., BOSTON.

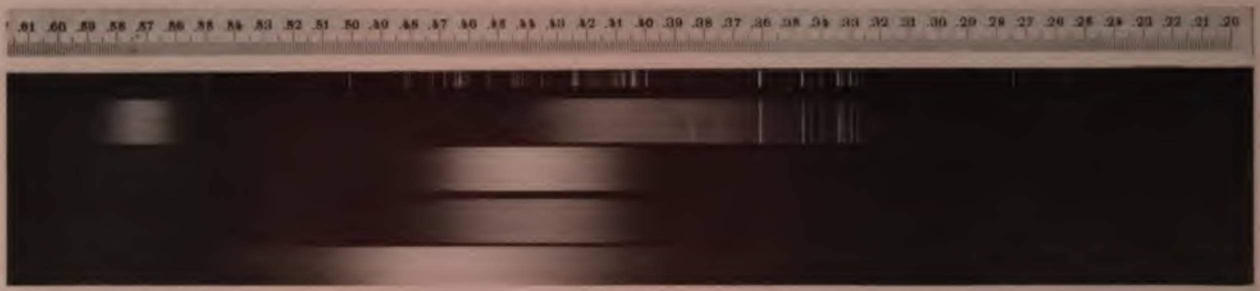


FIG. 90.

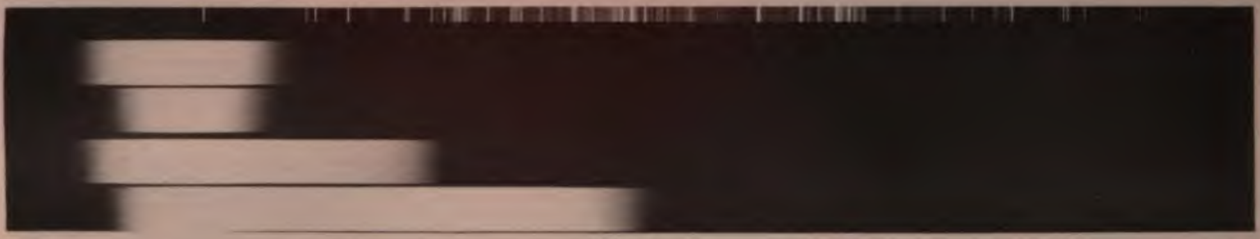


FIG. 91.

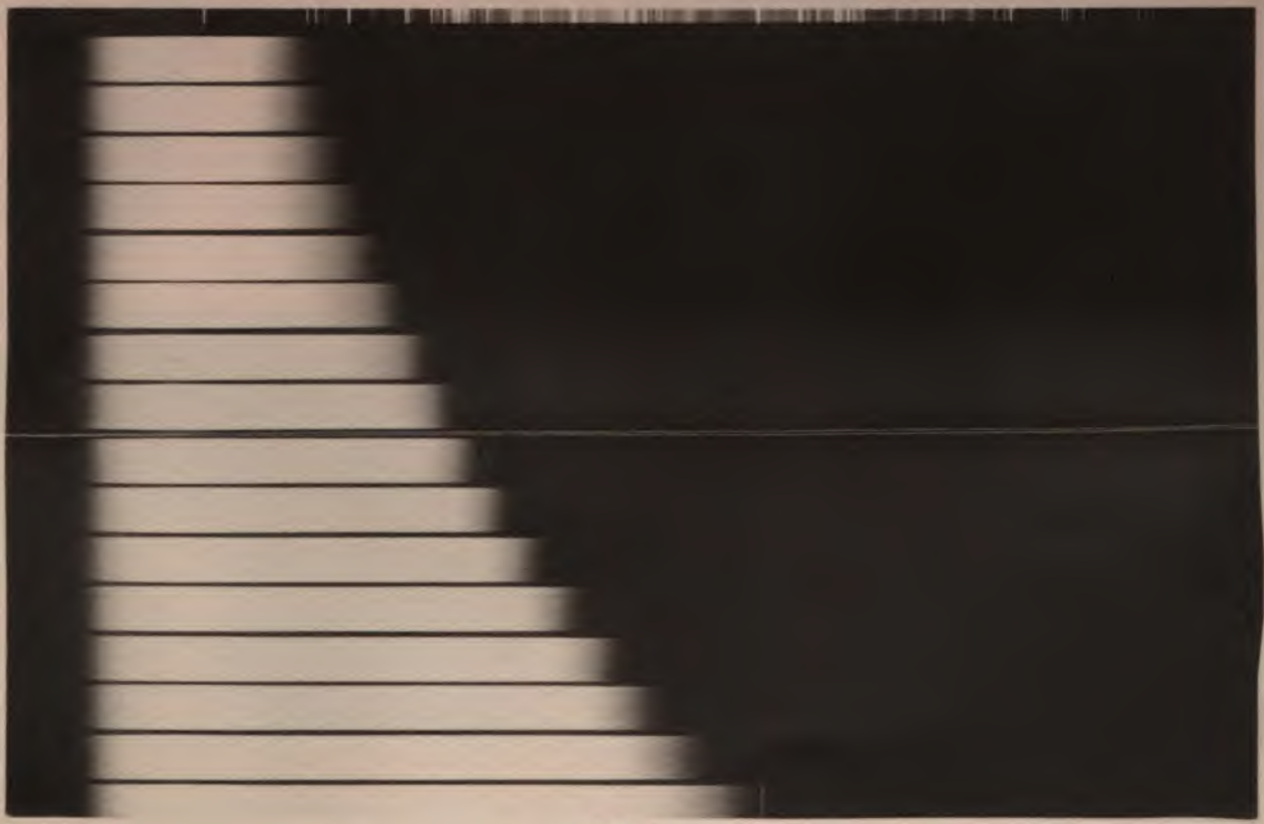


FIG. 92.

FIG. 90. SEE NO. 158. COBALT CHLORIDE IN SOLVENTS.
FIG. 91. SEE NO. 165. COPPER CHLORIDE IN SOLVENTS.
FIG. 92. SEE NO. 164. CHLORIDES OF CALCIUM AND COPPER.

HELIOTYPE CO., BOSTON.

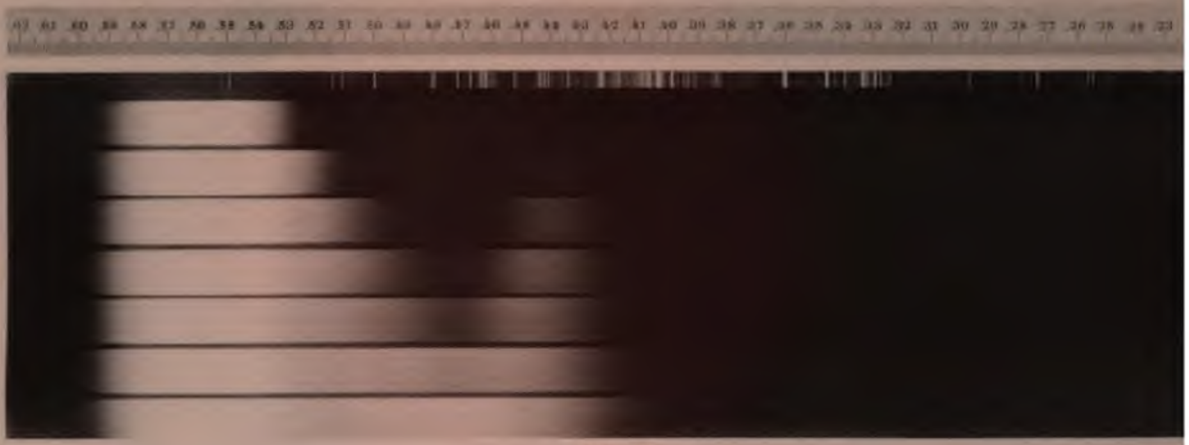


FIG. 93.

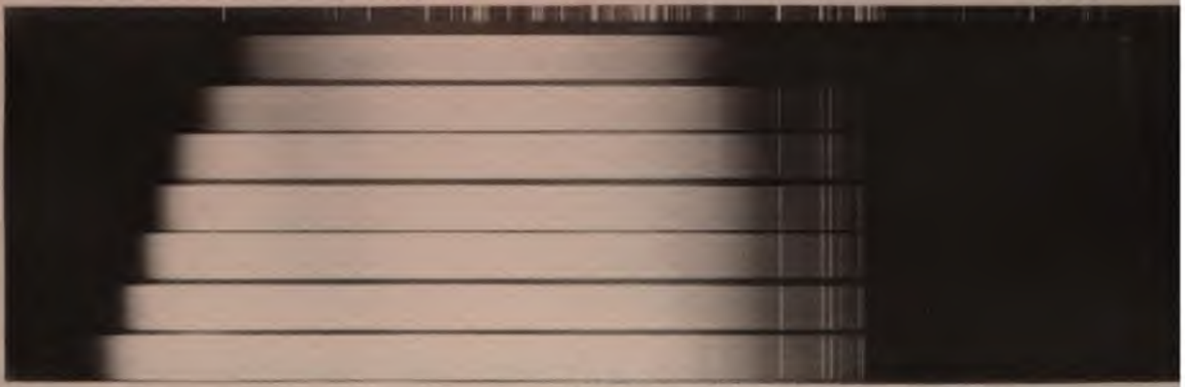


FIG. 94.

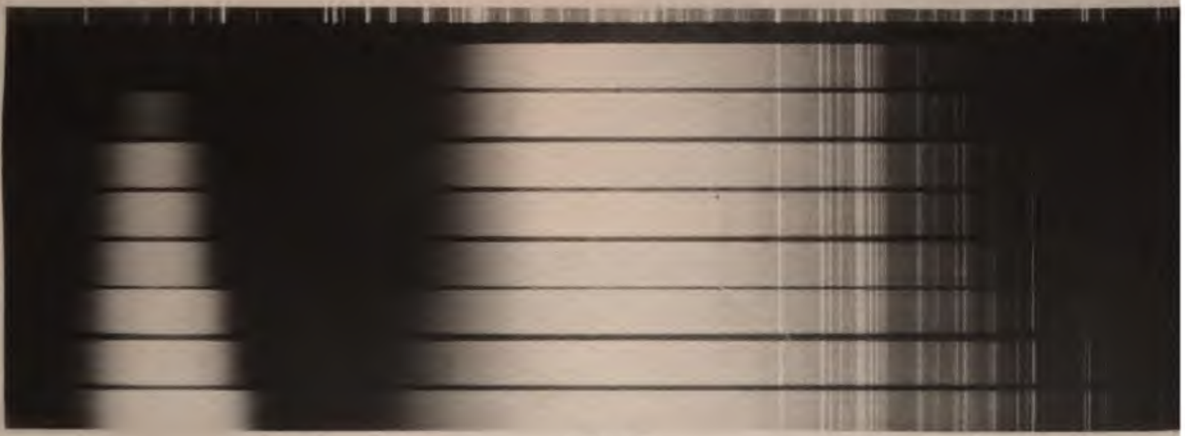


FIG. 95.

FIG. 93. SEE NO. 165. COPPER CHLORIDE IN ACETONE.
FIG. 94. SEE NO. 158. COBALT CHLORIDE IN ACETONE.
FIG. 95. SEE NO. 157. CHLORIDES OF ALUMINIUM AND COBALT.

HELIOTYPE CO., BOSTON.

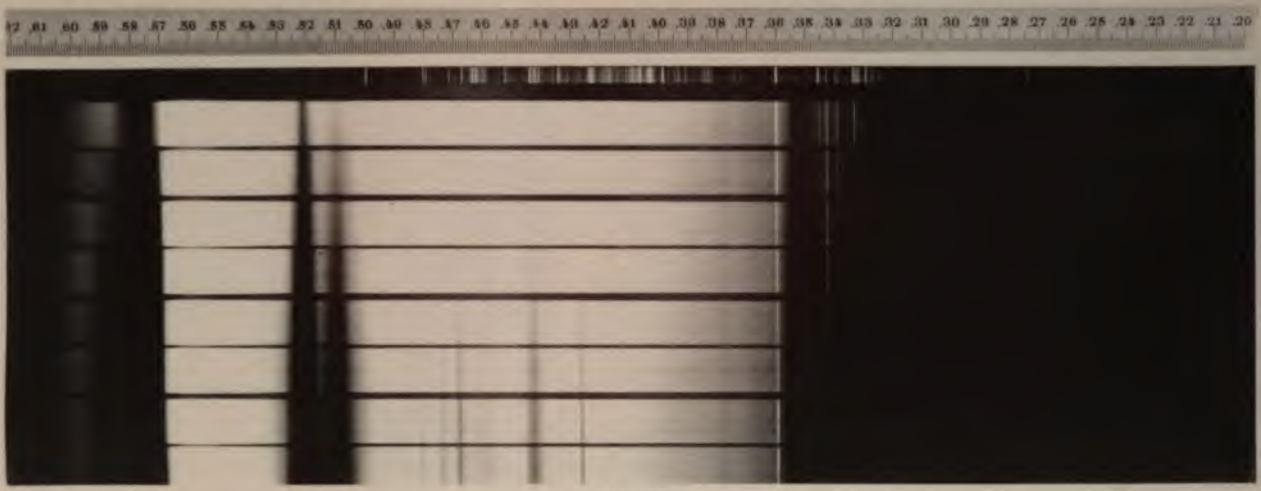


FIG. 96.

FIG. 97.

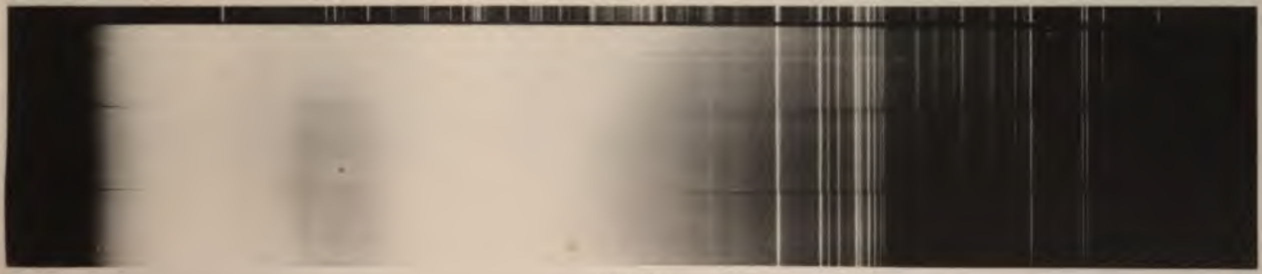


FIG. 98.

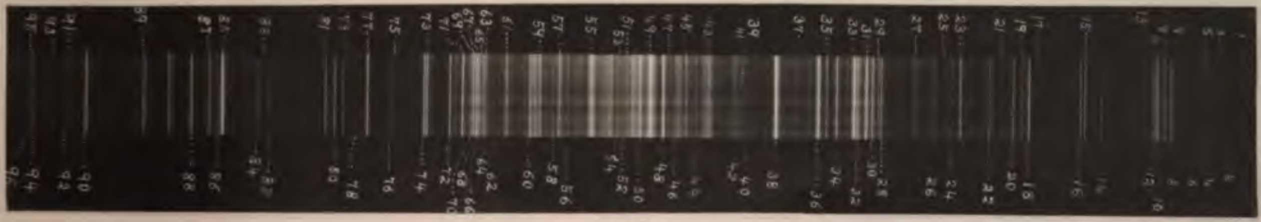


FIG. 99.

FIG. 96. SEE NO. 174. NEODYMIUM AMMONIUM NITRATE.
FIG. 97. SEE NO. 174. NEODYMIUM AMMONIUM NITRATE.
FIG. 98. SEE NO. 174. NEODYMIUM AMMONIUM NITRATE.
FIG. 99. SEE PAGE 169.

HELIOTYPE CO., BOSTON.

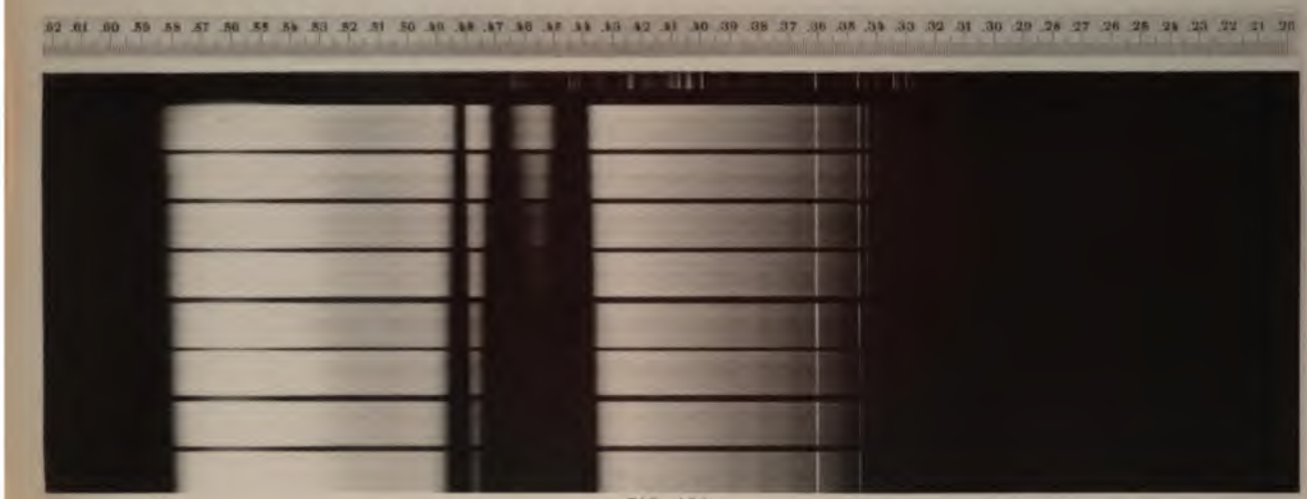


FIG. 100.

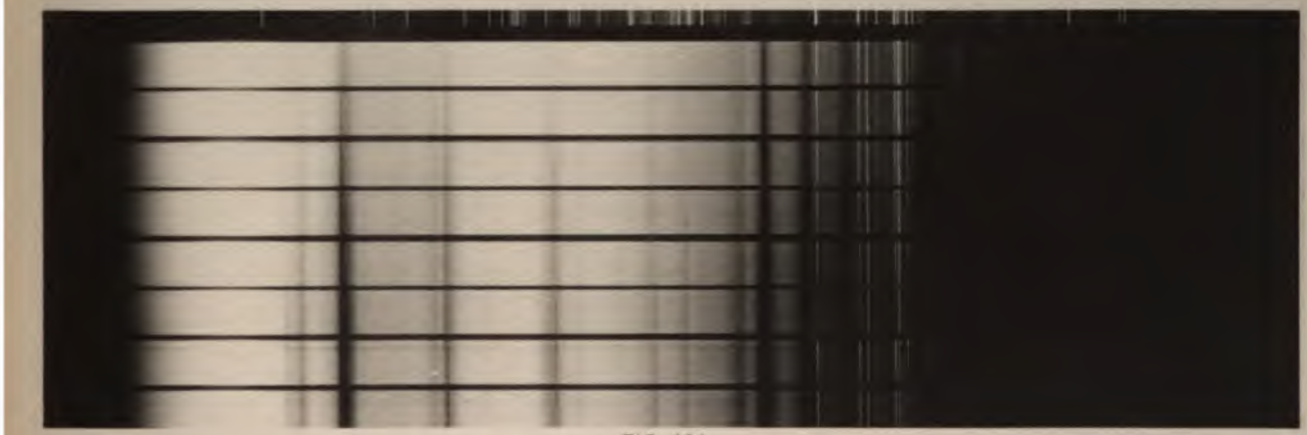


FIG. 101.

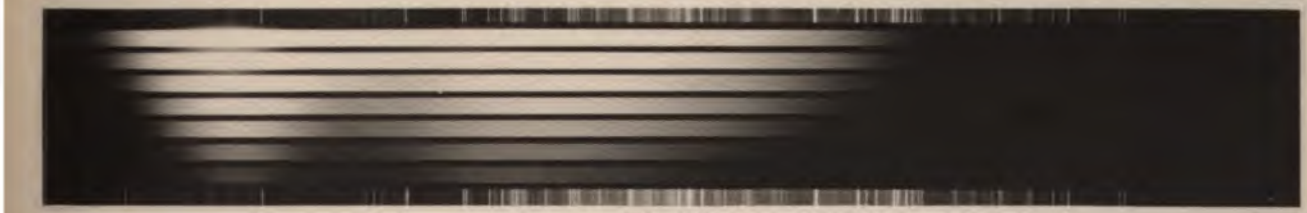


FIG. 102.

FIG. 100. SEE NO. 180. PRASEODYMIUM AMMONIUM NITRATE.
FIG. 101. SEE NO. 169. ERBIUM CHLORIDE.
FIG. 102. SEE PAGES 5 and 20

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INVESTIGATION OF INEQUALITIES IN THE MOTION OF THE MOON PRODUCED BY THE ACTION OF THE PLANETS

SIMON NEWCOMB

[illegible]

FRANK E. ROSS

[illegible]

INVESTIGATION OF INEQUALITIES IN THE MOTION OF THE MOON PRODUCED BY THE ACTION OF THE PLANETS

SIMON NEWCOMB

[illegible]

[illegible]

[illegible]
[illegible]
[illegible]

INVESTIGATION OF INEQUALITIES IN THE MOTION OF THE MOON PRODUCED BY THE ACTION OF THE PLANETS

BY

SIMON NEWCOMB

ASSISTED BY

FRANK E. ROSS

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PREFACE.

THE immediate incentive to the present work was the hope of explaining by gravitational theory the observed variations in the mean longitude of the Moon, shown by more than two centuries of observation to exist, but not yet satisfactorily accounted for. The author has published a number of papers and memoirs on this subject during the last forty years, terminating with a summary of the case, which appeared in the *Monthly Notices of the Royal Astronomical Society* for March, 1904. The deviations in question offer the greatest enigma yet encountered in explaining the motions of the heavenly bodies, and the present paper may be regarded as a contribution to the study of the problem thus offered.

While the work was in progress the completing chapter of Professor Brown's *Theory of the Moon's Motion* appeared. The actual work being based on Delaunay's theory, it seemed to be desirable to revise and correct it by Brown's results. In doing this the imperfections of Delaunay's theory as a basis became so evident, and the later theory proved to be so much better adapted to the purpose of the investigation, that the completed work gradually became step by step practically based upon Brown's theory, except in those parts requiring derivatives which could not be readily obtained except from Delaunay's literal expressions. Acknowledgment is due to Professor Brown for courteous advice and assistance which facilitated the use of his work for the purpose.

The theory of the action of the planets on the Moon being, in several points, the most intricate with which the mathematical astronomer has to deal, it is important that its development should be presented in a form to render as easy as possible the detection of errors or imperfections. In the arrangement of the work this end has been kept constantly in view. It is hoped that any investigator desiring to test the processes will find few difficulties except those necessarily inherent in the nature of the work.

To form a general conception of the arrangement it may be stated that the work naturally divides itself into four parts. One of these treats of the theory of the subject, including under this head not only the general equations, but the numerical details on which all the computations are based. In this part the fundamental quantities are reduced to products of two factors, one of which depends upon the coördinates of the planet; the other upon the geocentric coördinates of the Moon. The first factors, termed planetary, are numerically developed in Part II. This development falls into two parts, one treating the direct action of the planet, the other the indirect action through the Sun. In Part III is found the numerical development of the factors depending upon the Moon alone, and of their partial derivatives as to the lunar elements. In Part IV is presented the combinations of these two factors and the final results of the work.

A more complete summary in detail is found in the table of contents. An effort has been made to lessen the trouble of finding the definitions of the symbols used by collecting in the introduction definitions or references to these symbols as to the meaning of which doubt might be felt.

A word may be added as to the part taken by the author's assistant. At an early stage in the work Dr. Ross made a practically independent computation of the principal periodic inequalities, using the methods of Hill and Radau. In doing this he discovered the error of the Jovian evection as computed by them, which arose from the omission of what we may call the side-terms in the indirect action. His result for the coefficient was $1''.16$, in exact agreement with that originally found by Mr. Neville. In this early stage of the work the writer did not intend to do much more than revise these computations, and make a thorough investigation of the terms of long period. But he found the theory of the subject so interesting, and the opportunity for recasting the methods so attractive, that he was led to carry the work through, with Dr. Ross's assistance, on the basis of his own developments.

The next step in logical order is the rediscussion of the moon's mean longitude since 1650, as derived from occultations of stars, with a view of learning what modifications will be produced by the use of the more rigorous data now available, and the addition of thirty years to the period of available observations. This rediscussion will, the writer hopes, be his next contribution to the subject of the motion of the Moon.

It remains to add that the work has been prosecuted under the auspices of the Carnegie Institution of Washington, without the help of which it could not have been undertaken.

SIMON NEWCOMB.

WASHINGTON, MAY, 1907.

ACTION OF THE PLANETS ON THE MOON.

INTRODUCTION.

More than thirty years ago the author proposed to treat the action of the planets on the Moon by using the Lagrangian differential equations for the variation of the elements by considering as simultaneously variable, not only what are commonly called the elements of the Moon, but those of the orbit of the centre of mass of the Earth-Moon around the Sun also.[*] Twelve elements would thus come in, and the coördinates both of the Moon and of the Sun would be expressed in terms of the osculating values of all these elements.

Notwithstanding the favorable opinion of this method expressed at the time by Professor Cayley, and later, as to some of its processes, by Professor E. W. Brown, the author found that, in applying it unmodified, which he did during the years 1872–77, very long and complex computations were required in its application. The result was that the work, so far as it was carried, remained unpublished for nearly twenty years. Hoping that the general developments of the work and some of the details might be of use to subsequent investigators, the incomplete work was finally published in 1895.

About the same time with the publication of this work appeared the very elaborate one of Radau.[†] This work contains a seemingly exhaustive enumeration of possible inequalities of long period, and the numerical computation of a great number of lunar inequalities due to the action of the planets which had not previously been suspected.

On recommencing the work in 1904 it became very clear to the author that its completion by his former method, unmodified, would be impracticable, and that satisfactory results could best be reached by regarding the solar elements as constants, or known variables from the beginning. In the present investigation, therefore, the method has been modified so that the final values of the coördinates of the Moon, instead of being expressed as functions of the instantaneous elements of the Earth's disturbed motion, are expressed as functions of the mean elements. As thus modified it is substantially a continuation of that of Delaunay, as applied first by Hill and then by Radau. In this method the coördinates of the Sun, relative to the centre of gravity of the Earth and Moon, are regarded as known functions of the time. Then, when the action of the Sun alone is considered, the coördinates of the Moon relative to the Earth are found by the method of Delaunay, completed if necessary, as functions of six purely arbitrary constants.

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[*] *Liouville, Journal des Mathématiques*, 1871, March.

[†] *Annales de l'Observatoire de Paris.* [illegible] · XXI.

This solution of the problem of three bodies is supposed to be complete in advance. When the action of the planets is then taken into consideration, the only elements whose variations are to be determined by the Lagrangian equations are the six final elements of the Moon's motion. The variations in the coördinates of the Sun, due to the same action, are derived with great ease, and enter into the differential equations. In this way a system of six differential equations for the determination of the changes in the lunar elements is all that is necessary.

In setting forth the subject it is deemed unnecessary to repeat the derivation of the equations already found in astronomical literature. For this branch of the subject, reference may be had to Hill's paper in the American Journal of Mathematics, Vol. VI, and to Chapter XIII of the Treatise on the Lunar Theory by E. W. Brown. It is deemed necessary only to explain fully, at each point, the application of the method, and the meaning of the symbols introduced.

PART I.

DEVELOPMENT OF THE THEORY.

CHAPTER I.

FUNDAMENTAL DIFFERENTIAL EQUATIONS.

§ 1. *Notation.* The following notation is mostly used in this work:

G, when designating a point, centre of mass of Earth and Moon; m', mass of the Sun; m_2, mass of the Earth; m_3, mass of the Moon; m_4, mass of the Planet.

$$\mu = m_2 + m_3 \qquad \mu' = m' + \mu$$

x, y, z, r, geocentric coördinates and radius vector of the Moon, referred to the moving ecliptic;

x', y', z', r', coördinates and radius vector of the Sun, referred to the point G and the moving ecliptic;

ξ, η, ζ, and ρ, the ratios of x, y, z, and r of the Moon to the mean distance of the latter: $x = a\xi$, etc. When unmarked the coördinates are referred to a moving X-axis directed toward the mean Sun;

x_1, y_1, Moon coördinates referred to the mean Moon as the X-axis;

Δ, distance of the Planet from G;

S, cosine of angle between r and r';

S', cosine of angle between r and Δ;

P_0, potential function of mutual action of Earth and Moon;

Ω, potential function for action of Sun on Moon;

R, potential function for action of Planet on Moon;

l, π, θ, mean longitude, longitude of perigee, longitude of node of Moon;

π_1, θ_1, motions of π and θ in unit of time (quantities of dimensions T^{-1});

N, motion of argument in unit of time;

n, ratio of motion of an argument to n, the mean motion of the Moon;

ν, the integrating factor, generally $= n/\mathrm{N}$;

a, e, g, defined in (43), § 22;

K, C, D, planetary coefficients for the direct action, defined in § 20;

p, q, κ_4, lunar coefficients, § 20 Eq. (36);

G, J, I, planetary coefficients for the indirect action, defined in § 24;

G is also used for a combined lunar and planetary argument;

$\mathfrak{a}$, logarithm of a, the Moon's mean distance;

V, M, J, S, the mean longitudes of the respective planets, Venus, Mars, Jupiter, and Saturn measured, in each case, from the Earth's perihelion: π' for 1800=99.°5.

The abbreviation "Action" has been used to designate the previous work of the author on this subject—"Theory of the Inequalities in the motion of the Moon produced by the Action of the Planets"; forming Part III of Astronomical Papers of the American Ephemeris, Vol. V.

§ 2. *Dimensions of quantities.* In this subject it will be found helpful to the reader and investigator to have, in the case of the principal equations, a statement of their dimensions in terms of the fundamental units of Mass, Time, and Length. In strictness an independent unit of mass is not necessary in gravitational astronomy, because the most convenient unit is that mass which, on an equal mass at unit distance, exerts a unit force of gravitation. But it is still sometimes convenient to use this unit in the equations, although it is a derived one.

In the case of each system of equations which are regarded as fundamental will be found the dimensions of the terms which form its members, the signification being as follows:

T, Time; L, Length; M, Mass.

The definition of the unit of mass just given leads to the relation

$$M = L^3T^{-2}$$

In this way it will be much easier than it would be without this help to appreciate the degree of magnitude of small quantities. Considered by itself, no concrete quantity can be regarded as small or great; it is so only when compared with other quantities of the same kind, or, to speak more accurately, of the same dimensions in fundamental quantities. The ratios of two fundamental quantities of the same kind are pure numbers, and these may be large or small to any extent.

§ 3. *Fundamental differential equations.*

Putting

x_1, y_1, z_1, the geocentric coördinates of the Moon referred to any system of fixed axes,

P, the total potential

the differential equations to be integrated may be written

$$D_t^2x_1 = \frac{\partial P}{\partial x_1} \qquad D_t^2y_1 = \frac{\partial P}{\partial y_1} \qquad D_t^2z_1 = \frac{\partial P}{\partial z_1} \quad [\text{Dimensions} = ML^{-2} = LT^{-2}] \qquad (1)$$

§ 4. *Transformation to the moving ecliptic.* In the preceding equations the coördinates are referred to fixed axes. In astronomical practice the coördinates of the heavenly bodies are referred to the moving ecliptic. The latter carries the plane of the Moon's orbit with it in its motion. It therefore seems desirable to refer the motion, in the first place, to the moving ecliptic.

To do this let us put

x, y, z, coördinates referred to the moving ecliptic;

κ, the speed of motion of the plane of the ecliptic;

Π, the longitude of the ascending node of the moving on the fixed ecliptic, or of the instantaneous axis of rotation of the ecliptic. At the present time we have $\Pi = 173^\circ$, nearly.

Then, regarding κt as infinitesimal, the expression for the moving coördinates in terms of the fixed ones will be

$$
\begin{aligned}
x &= x_1 - z_1 t\kappa \sin \Pi \\
y &= y_1 + z_1 t\kappa \cos \Pi \\
z &= z_1 + x_1 t\kappa \sin \Pi - y_1 t\kappa \cos \Pi
\end{aligned}
$$

Putting for brevity

$$p = \kappa \sin \Pi \qquad q = \kappa \cos \Pi \qquad [\text{Dim. of } p, q, \text{ and } \kappa = T^{-1}],$$

these expressions become

$$
\begin{aligned}
x &= x_1 - pz_1 t \\
y &= y_1 + qz_1 t \\
z &= z_1 + px_1 t - qy_1 t
\end{aligned}
\tag{2}
$$

Differentiating them twice as to the time, regarding p and q as constant, we have

$$
\begin{aligned}
D_t^2 x &= D_t^2 x_1 - ptD_t^2 z_1 - 2pD_t z_1 \\
D_t^2 y &= D_t^2 y_1 + qtD_t^2 z_1 + 2qD_t z_1 \\
D_t^2 z &= D_t^2 z_1 + ptD_t^2 x_1 - qtD_t^2 y_1 + 2pD_t x_1 - 2qD_t y_1
\end{aligned}
\tag{3}
$$

Regarding P, originally a function of x_1, y_1, and z_1, as becoming a function of x, y, and z through the substitution (2) we have

$$\frac{\partial P}{\partial x_1} = \frac{\partial P}{\partial x} + pt\frac{\partial P}{\partial z} \qquad \frac{\partial P}{\partial y_1} = \frac{\partial P}{\partial y} - qt\frac{\partial P}{\partial z} \qquad \frac{\partial P}{\partial z_1} = \frac{\partial P}{\partial z} - pt\frac{\partial P}{\partial x} + qt\frac{\partial P}{\partial y}$$

Substituting these expressions for $D_t^2 x_1$, $D_t^2 y_1$, and $D_t^2 z_1$ in (3) and dropping terms of the second order in pt and qt we find

$$
\begin{aligned}
D_t^2 x &= \frac{\partial P}{\partial x} - 2pD_t z_1 \\
D_t^2 y &= \frac{\partial P}{\partial y} + 2qD_t z_1 \\
D_t^2 z &= \frac{\partial P}{\partial z} + 2pD_t x_1 - 2qD_t y_1
\end{aligned}
\tag{4}
$$

Equations of this form were used by Hill for the same purpose.*

---

* *Annals of Mathematics*, vol. 1, 1890.

It follows that if we add to P the terms

$$\Delta R = 2p(zD_t x_1 - xD_t z_1) + 2q(yD_t z_1 - zD_t y_1) \tag{5}$$

so that the potential shall become

$$P + \Delta R \qquad [\text{Dim.} = ML^{-1} = L^2 T^{-2}]$$

the fundamental differential equations in x, y, and z, will retain the form (1) unchanged, and the coördinates referred to the moving ecliptic will be determined by the general equations

$$D_t^2 x = \frac{\partial P}{\partial x} \qquad D_t^2 y = \frac{\partial P}{\partial y} \qquad D_t^2 z = \frac{\partial P}{\partial z} \tag{6}$$

In ΔR the symbols x_1, y_1, and z_1 have the same meanings as x, y, and z, but they are to be regarded as constant when ΔR is differentiated as to the lunar elements.

§ 5. *Preliminary form of the potential function.*

We put Ω for the part of the potential P due to the action of the Sun. This part is developed in a series proceeding according to the powers of r/r' in the well-known form

$$\Omega = \frac{m'r^2}{2r'^3}(3S^2 - 1) + \frac{m'r^3}{2r'^4} \cdot \frac{m_2 - m_3}{\mu}(5S^3 - 3S) + \cdots \tag{7}$$

where S, the cosine of the angle between the radii vectores of the Moon and Sun from the point G, is determined by the equation

$$rr'S = xx' + yy' + zz'$$

When we assign to x', y', z', and r' their elliptic values, we have what may be called the Delaunay part of the potential. We put

Ω_0, the Delaunay part of Ω.
Ω_p, the increment of Ω_0 produced by the action of the planets on the Earth.

The part R of P, due to the direct action of the planet in changing the coördinates of the Earth relative to the Moon, may be formed from Ω in (7) by replacing

$$m', \; r', \; x', \; y', \text{ and } z'$$

by

$$m_4, \; \Delta, \; X, \; Y, \text{ and } Z$$

where m_4 is the mass of the planet, and Δ, X, Y, and Z its distance and coördinates relative to the point G. Putting R for this part we have for its principal term

$$R = \frac{m_4 r^2}{2\Delta^3}(3S'^2 - 1) \tag{7a}$$

where S' is determined by the equation

$$r\Delta S' = (x' + x_4)x + (y' + y_4)y + (z' + z_4)z$$

x_4, y_4, and z_4 being the heliocentric coördinates of the planet.

We have thus separated the potential of all the actions changing the coördinates of the Moon relative to the Earth into the following five parts.

A. The part generated by the mutual action of the Earth and Moon, $P_0 = \mu/r$, which taken alone would give rise to an undisturbed elliptic motion of the Moon around the Earth.

B. The part Ω_0 generated by the action of the Sun, assuming the point G to move in an elliptic orbit.

C. The part Ω_p, the increment of Ω_0 due to the action of the planets on the point G.

D. The part R due to the direct action of the planet. Developed in the same way as the highest term of Ω the principal term of this part is formed from Ω by replacing m', x', y', and z' by the mass and G-coördinates of the planet. The value of its principal term is given in (7*a*).

E. The part ΔR arising from the reference of the coördinates to the moving ecliptic.

The complete value of P thus becomes

$$P = P_0 + \Omega_0 + \Omega_p + R + \Delta R \tag{8}$$

and we are to consider this expression as replacing P in the equations (6).

§ 6. ***Reduction of the terms of the potential function for the indirect action.***

By substituting for S in (7) its value, the first and principal term of Ω becomes a linear function of the six squares and products of the lunar coördinates x, y, and z, which we may write

$$\Omega = T_1x^2 + T_2y^2 + T_3z^2 + 2T_4xy + 2T_5xz + 2T_6yz \tag{9}$$

Moreover, since we form the part Ω_p of the potential by assigning increments to T, and the part R by making T a function of the elements of the planet, it follows that both of these parts as well as Ω are of this same form.

For the first and principal term of Ω_0 in which the higher powers of r/r' are dropped we have

$$\begin{aligned}
T_1 &= \frac{m'}{r'^3}\left(\frac{3}{2}\frac{x'^2}{r'^2} - \frac{1}{2}\right) & T_4 &= \frac{3}{2}\frac{m'}{r'^3}\cdot\frac{x'y'}{r'^2} \\
T_2 &= \frac{m'}{r'^3}\left(\frac{3}{2}\frac{y'^2}{r'^2} - \frac{1}{2}\right) & T_5 &= \frac{3}{2}\frac{m'}{r'^3}\cdot\frac{x'z'}{r'^2} \qquad [\text{Dim.} = ML^{-3}] \\
T_3 &= \frac{m'}{r'^3}\left(\frac{3}{2}\frac{z'^2}{r'^2} - \frac{1}{2}\right) & T_6 &= \frac{3}{2}\frac{m'}{r'^3}\cdot\frac{y'z'}{r'^2}
\end{aligned} \tag{10}$$

The study of the second term, which it may be advisable to examine for sensible results, is postponed, and Ω is taken as equal to its principal part. The value of Ω_p is then found by adding to the preceding values of T_i their increments produced by the action of the planets upon the coördinates x', y', and z' of the Sun. If we put

v', the longitude of the Sun

and take the moving ecliptic as the plane of reference, we may regard z', the periodic perturbations of the latitude, as infinitesimal and write

$$x' = r' \cos v' \qquad y' = r' \sin v' \qquad z' = r' \sin \beta'$$

where β' is the Sun's latitude, a minute purely periodic quantity.

Substituting these values in (10), the expressions for the coefficients T become

$$\begin{aligned} T_1 &= \frac{m'}{r'^3}(\tfrac{1}{4} + \tfrac{3}{4} \cos 2v') & T_2 &= \frac{m'}{r'^3}(\tfrac{1}{4} - \tfrac{3}{4} \cos 2v') & T_3 &= -\frac{m'}{2r'^3} \\ T_4 &= \frac{3}{4}\frac{m'}{r'^3} \sin 2v' & T_5 &= \frac{3}{2}\frac{m' \sin \beta' \cos v'}{r'^3} & T_6 &= \frac{3}{2}\frac{m' \sin \beta' \sin v'}{r'^3} \end{aligned} \tag{10a}$$

If we assign to these quantities their elliptic values, (7) will become Ω_0 for which the integration is assumed in advance. We have now to assign to v' and r' the increments $\delta v'$ and $r'\delta\rho'$, ρ' being the Naperian logarithm of r'. The resulting increments of the coefficients are

$$\begin{aligned} \delta T_1 &= \frac{3m'}{4r'^3}\{-2 \sin 2v'\delta v' - 3 \cos 2v'\delta\rho' - \delta\rho'\} & \delta T_2 &= \frac{3m'}{4r'^3}\{2 \sin 2v'\delta v' + 3 \cos 2v'\delta\rho' - \delta\rho'\} \\ \delta T_3 &= \frac{3m'}{2r'^3}\delta\rho' & \delta T_4 &= \frac{3m'}{4r'^3}\{2 \cos 2v'\delta v' - 3 \sin 2v'\delta\rho'\} \end{aligned} \tag{11}$$

The values of δT_5 and δT_6 will be the original values (9) of T_5 and T_6 as they are due wholly to the action of the planet. With them the expression for Ω_p derived from (9) becomes

$$\Omega_p = \delta T_1 x^2 + \delta T_2 y^2 + \delta T_3 z^2 + 2\delta T_4 xy + 2 T_5 xz + 2 T_6 yz \tag{12}$$

§ 7. *Reduction of R, the potential of direct action.*

By substituting for S' in the principal term (7a) of R its expression in terms of the G-coördinates of the planet we shall have

$$R = \tfrac{3}{2} m_4 R_2 \tag{13}$$

where

$$R_2 = Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz \qquad [\text{Dim.} = L^{-1}]$$

the values of the coefficients being

$$\begin{aligned} A &= \frac{(x' + x_4)^2}{\Delta^5} - \frac{1}{3}\frac{1}{\Delta^3} & D &= \frac{(x' + x_4)(y' + y_4)}{\Delta^5} \\ B &= \frac{(y' + y_4)^2}{\Delta^5} - \frac{1}{3}\frac{1}{\Delta^3} & E &= \frac{(x' + x_4)(z' + z_4)}{\Delta^5} \qquad [\text{Dim.} = L^{-3}] \quad (14) \\ C &= \frac{(z' + z_4)^2}{\Delta^5} - \frac{1}{3}\frac{1}{\Delta^3} & F &= \frac{(y' + y_4)(z' + z_4)}{\Delta^5} \end{aligned}$$

It should be noted that these coefficients require the factor $\frac{3}{2}$ to make them directly comparable with T_1, T_2, etc., in (10).

§ 8. *Complete form of the fundamental equations.*

Comparing the expressions (12) to (14) we see that Ω_p and R are of the same form, and that the principal terms of each are products of two factors, of which one depends solely on the heliocentric coördinates of the Sun and planet, and the other is a square or product of the coördinates of the Moon. Moreover, if we put, for brevity,

$$P_1 = \Omega_p + R + \Delta R \qquad (15)$$

the fundamental differential equations may be written

$$D_t^2 x = \frac{\partial(P_0 + \Omega_0)}{\partial x} + \frac{\partial P_1}{\partial x} \qquad D_t^2 y = \frac{\partial(P_0 + \Omega_0)}{\partial y} + \frac{\partial P_1}{\partial y} \qquad D_t^2 z = \frac{\partial(P_0 + \Omega_0)}{\partial z} + \frac{\partial P_1}{\partial z} \qquad (16)$$

where x, y, and z are coördinates referred to the moving ecliptic as the fundamental plane.

We shall now consider these differential equations as solved for the case when P_1 is dropped from the second members. The problem will then be that of the solution when P_1 is included; and this problem will be attacked by the Lagrangian method of variation of elements.

CHAPTER II.

DEVELOPMENT AND INTEGRATION OF THE DIFFERENTIAL EQUATIONS FOR THE VARIATION OF THE ELEMENTS.

§ 9. The problem being to integrate equations (16), we shall regard as known quantities the coördinates x', y', z' of the Sun, which enter implicitly into the equations, as well as those of the planets relative to the Sun. The problem then is to express the values of x, y, and z in terms of the fundamental constants implicitly contained in the differential equations, and six other arbitrary constants which we regard as elements of the Moon's motion.

The solution of the equations is separated into two parts by applying the Lagrangian method of the variation of elements. We have first the Delaunay solution, in which P_1 is dropped. This solution gives the orbit of the Moon around the Earth under the influence of the Sun's and Earth's attraction alone. From it we are to pass, by the method of variation of elements, to a solution when P_1 is taken account of.

We accept the results of Delaunay, as found in his work, as forming the basis of the first solution, the results needing only certain modifications in the terms depending on the Sun's parallax, arising from the fact that he did not take into account the mass of the Moon, and certain reductions, to reduce them to the required form. This being done we have values of the Moon's coördinates satisfying the differential equations in the case $P=\mu/r+\Omega_0$ and expressed as functions of six arbitrary constants

$$a,\ e,\ \gamma,\ l_0,\ \pi_0,\ \theta_0$$

and of the time t. The latter enters only through the quantities l, π, and θ, named and defined thus

Mean longitude: $l=\iota_0+nt$ Long. of perigee: $\pi=\pi_0+\pi_1 t$ Long. of node: $\theta=\theta_0+\theta_1 t$ (17)

where n, π_1, and θ_1 are functions of a, e, and γ.

I use the quantities l, π, and θ instead of Delaunay's l, g, and h, which are the mean anomaly, the angle node to perigee, and the longitude of the node. The expressions for the symbols used here in terms of those used by Delaunay are therefore

$l =$ Delaunay's $h+g+l$ $\pi =$ Delaunay's $h+g$ $\theta =$ Delaunay's h (18)

The fundamental idea of the Lagrangian method, which we propose to apply to the present problem, is that the six arbitrary elements are to become such functions of the time that the solution which satisfies (16) when $P_1 = 0$ shall still satisfy it when the variable values of the elements are substituted for the constant values in the expressions for the coördinates. The derivatives of the elements as to the time may be formed by known processes, but the details of these processes are unnecessary, because Delaunay gives their results in a form most convenient for our purpose.

§ 10. *Canonical form of the differential equations.*

We see from (5), (12), (13), and (15) that P_1 is a function of given quantities and of the Moon's coördinates. By substituting for the latter their expressions in terms of the six arbitrary constants of the first integration, P_1 becomes a function of a, e, γ, l, π, and θ. The differential variations of the elements are then expressed in the most condensed form by replacing a, e, and γ by three other quantities c_1, c_2, and c_3, functions of a, e, γ, so chosen that the differential equations to be solved shall be

$$\begin{aligned} D_t c_1 &= \frac{\partial P_1}{\partial l} & D_t l_0 &= -\frac{\partial P_1}{\partial c_1} & & \\ D_t c_2 &= \frac{\partial P_1}{\partial \pi} & D_t \pi_0 &= -\frac{\partial P_1}{\partial c_2} & &[\text{Dim.} = ML^{-1}] \qquad (19) \\ D_t c_3 &= \frac{\partial P_1}{\partial \theta} & D_t \theta_0 &= -\frac{\partial P_1}{\partial c_3} & & \end{aligned}$$

The variable elements c_1, c_2, and c_3 are functions of Delaunay's L, G, H.

$$c_1 = L \qquad \underset{[\text{Dim.} = L^{\frac{1}{2}}M^{\frac{1}{2}} = L^2T^{-1}]}{c_2 = G - L} \qquad c_3 = H - G \qquad (20)$$

§ 11. *Transformation of the canonical elements.*

The canonical elements c_1, c_2, and c_3 can not be used explicitly in the processes of solution. We have therefore to express them in terms of a, e, and γ. The values of L, G, and H are not given by Delaunay in terms of the final a, e, and γ, but of preliminary ones from which the required expressions are to be derived as follows:

1. In Vol. II, pp. 235–236, Delaunay gives the expressions for L, G, and H in terms of the a, e, and γ which resulted immediately from his processes of integration.

2. On p. 800 he gives the transformation of these a, e, γ, into the final values of these quantities which appear in the expression for the Moon's coördinates, which are those we are to use.

To find from these data the expressions for the derivatives of L, G, H in terms of the final a, e, γ, I shall write a, e, g, n, for the quantities a, e, γ, n, as found on pp. 235–236 of Delaunay, Vol. II, and shall also put

$$\mathrm{m} = \frac{n'}{\mathrm{n}}$$

The forms which we have to use are:

$$L,\ G,\ H = f(\mathrm{a},\ \mathrm{e},\ \mathrm{g},\ \mathrm{m}) \qquad \mathrm{a},\ \mathrm{e},\ \mathrm{g} = f(a,\ e,\ \gamma,\ m) \tag{21}$$

Noticing that m is a function of a and m of a, we shall then have

$$\frac{\partial L}{\partial a} = \left(\frac{\partial L}{\partial \mathrm{a}} + \frac{\partial L}{\partial \mathrm{m}}\,\frac{\partial \mathrm{m}}{\partial \mathrm{a}}\right)\frac{\partial \mathrm{a}}{\partial a} + \frac{\partial L}{\partial \mathrm{e}}\,\frac{\partial \mathrm{e}}{\partial a} + \frac{\partial L}{\partial \mathrm{g}}\,\frac{\partial \mathrm{g}}{\partial a} \tag{22}$$

with similar forms for G and H.

§ 12. *Form of the partial derivatives.* Two points in the use of the partial derivatives are these:

α. In taking the partial derivatives I use the logarithm of a and of a instead of these quantities as the variables with respect to which derivatives are to be formed. Homogeneity in the equations is thus secured, the variables being all pure numbers, or quantities of dimensions o. We put

$$\alpha = \log a \qquad \text{whence} \qquad a = e^{\alpha}$$

β. The quantities n and n are defined as functions of a and of a respectively by the equations

$$\mathrm{a}^3\mathrm{n}^2 = a^3 n^2 = \mu$$

It follows that if we have an expression M developed in powers of m or m,

$$M = a^i(M_0 + M_1 m + M_2 m^2 + \cdots)$$

we shall have

$$\frac{\partial M}{\partial \alpha} = a^i\left(iM_0 + \left(i + \tfrac{3}{2}\right)M_1 m + \left(i + \tfrac{6}{2}\right)M_2 m^2 + \cdots\right) \tag{23}$$

§ 13. *Numerical values of the fundamental quantities.*

Instead of effecting the preceding transformations analytically, to put the equations (21) into numbers, we use the numerical values of e, γ, and m given by Delaunay in his Vol. II, pp. 801–802, namely

$$e = .054\,8993 \qquad \gamma = .044\,8866 \qquad m = .074\,8013 \tag{24}$$

We then find from his expressions on p. 800

$$\mathrm{a} = 0.996\,493a = [9.998\,474]a$$

$$\mathrm{an} = 1.001\,758an = [0.000\,763]an$$

$$\mathrm{a}^2\mathrm{n} = 0.998\,245a^2n = [9.999\,237]a^2n$$

$$\mathrm{m} = 0.994\,743m = [9.997\,711]m = 0.074\,4082$$

$$\mathrm{e} = 0.054\,867 \qquad \mathrm{g} = 0.044\,993$$

We also find, from these numbers, the following values of the required partial derivatives for the numerical transformation

$$\frac{\partial \mathrm{a}}{\partial a} = 0.986\,691a \qquad \frac{\partial \mathrm{e}}{\partial a} = -0.007\,37e = -0.000\,404 \qquad \frac{\partial \mathrm{g}}{\partial a} = +0.006\,85\gamma = 0.000\,308$$

$$\frac{\partial \mathrm{a}}{\partial e} = -0.001\,375a \qquad \frac{\partial \mathrm{e}}{\partial e} = +0.999\,61 \qquad \frac{\partial \mathrm{g}}{\partial e} = +0.000\,202$$

$$\frac{\partial \mathrm{a}}{\partial \gamma} = +0.001\,353a \qquad \frac{\partial \mathrm{e}}{\partial \gamma} = -0.001\,22e = -0.000\,067 \qquad \frac{\partial \mathrm{g}}{\partial \gamma} = +1.002\,324$$

Then, from Delaunay, II, p. 236, we find

$$L = 1.000\,197\mathrm{a}^2\mathrm{n} \qquad G = 0.998\,586\mathrm{a}^2\mathrm{n} \qquad H = 0.994\,549\mathrm{a}^2\mathrm{n}$$

$$\mathrm{a}\frac{\partial L}{\partial \mathrm{a}} = 0.500\,158\mathrm{a}^2\mathrm{n} \qquad \mathrm{a}\frac{\partial G}{\partial \mathrm{a}} = 0.499\,697\mathrm{a}^2\mathrm{n} \qquad \mathrm{a}\frac{\partial H}{\partial \mathrm{a}} = 0.497\,696\mathrm{a}^2\mathrm{n}$$

$$\frac{\partial L}{\partial \mathrm{e}} = -0.000\,088\mathrm{a}^2\mathrm{n} \qquad \frac{\partial G}{\partial \mathrm{e}} = -0.052\,410\mathrm{a}^2\mathrm{n} \qquad \frac{\partial H}{\partial \mathrm{e}} = -0.052\,185\mathrm{a}^2\mathrm{n}$$

$$\frac{\partial L}{\partial \mathrm{g}} = -0.000\,007\mathrm{a}^2\mathrm{n} \qquad \frac{\partial G}{\partial \mathrm{g}} = -0.000\,035\mathrm{a}^2\mathrm{n} \qquad \frac{\partial H}{\partial \mathrm{g}} = -0.179\,474\mathrm{a}^2\mathrm{n}$$

§ 14. *Formation of the transformed differential equations.*

Let us now return to the equations (19), in which we have to replace c_1, c_2, and c_3 by a, e, and γ. We have, for any c,

$$\frac{dc}{dt} = \frac{dc}{d\mathrm{a}}\frac{d\mathrm{a}}{dt} + \frac{dc}{d\mathrm{e}}\frac{d\mathrm{e}}{dt} + \frac{dc}{d\mathrm{g}}\frac{d\mathrm{g}}{dt}$$

and

$$\frac{dc}{da} = \frac{dc}{d\mathrm{a}}\frac{d\mathrm{a}}{da} + \frac{dc}{d\mathrm{e}}\frac{d\mathrm{e}}{da} + \frac{dc}{d\mathrm{g}}\frac{d\mathrm{g}}{da}$$

In the case of c_1 we have from (20)

$$\frac{dc_1}{da} = \frac{dL}{da}$$

so that the numerical expressions need not be repeated. For the derivatives of c_2 and c_3 we find

$$\mathrm{a}\frac{\partial c_2}{\partial \mathrm{a}} = -0.000\,461\mathrm{a}^2\mathrm{n} \qquad \frac{\partial c_2}{\partial \mathrm{e}} = -0.052\,322\mathrm{a}^2\mathrm{n} \qquad \frac{\partial c_2}{\partial \mathrm{g}} = -0.000\,028\mathrm{a}^2\mathrm{n}$$

$$\mathrm{a}\frac{\partial c_3}{\partial \mathrm{a}} = -0.002\,001\mathrm{a}^2\mathrm{n} \qquad \frac{\partial c_3}{\partial \mathrm{e}} = +0.000\,225\mathrm{a}^2\mathrm{n} \qquad \frac{\partial c_3}{\partial \mathrm{g}} = -0.179\,436\mathrm{a}^2\mathrm{n}$$

By substitution in the form (22) we now find

$$\begin{array}{lll}
\frac{\partial c_1}{\partial \alpha} = 0.494\,369a^2n & \frac{\partial c_1}{\partial e} = -0.000\,777a^2n & \frac{\partial c_1}{\partial \gamma} = 0.000\,671a^2n \\
\frac{\partial c_2}{\partial \alpha} = -0.000\,435a^2n & \frac{\partial c_2}{\partial e} = -0.052\,209a^2n & \frac{\partial c_2}{\partial \gamma} = -0.000\,025a^2n \\
\frac{\partial c_3}{\partial \alpha} = -0.002\,033a^2n & \frac{\partial c_3}{\partial e} = 0.000\,191a^2n & \frac{\partial c_3}{\partial \gamma} = -0.179\,538a^2n
\end{array} \tag{25}$$

We now have the data for transforming the equations (19), p. 14, so as to express the differential variations of α, e, and γ instead of c_1, c_2, and c_3, and to express those of l_0, π_0, and θ_0 in terms of the partial derivatives of R as to α, e, and γ. For this purpose we need the nine partial derivatives of α, e, and γ as to c_1, c_2, and c_3. We shall express these nine derivatives by means of the nine numerical factors

$$\alpha_i, \; e_i, \; \gamma_i, \cdots (i = 1:2:3)$$

defined by the equations

$$\alpha_i = a^2n\frac{\partial \alpha}{\partial c_i} \qquad e_i = a^2n\frac{\partial e}{\partial c_i} \qquad \gamma_i = a^2n\frac{\partial \gamma}{\partial c_i}$$

The numerical values of these coefficients are most expeditiously found in the following way. Multiplying the first three equations (19) in order by the respective factors

$$\frac{\partial \alpha}{\partial c_1}, \quad \frac{\partial \alpha}{\partial c_2}, \quad \text{and} \quad \frac{\partial \alpha}{\partial c_3}$$

we have

$$D_t\alpha = \frac{\partial \alpha}{\partial c_1}\frac{\partial P_1}{\partial l} + \frac{\partial \alpha}{\partial c_2}\frac{\partial P_1}{\partial \pi} + \frac{\partial \alpha}{\partial c_3}\frac{\partial P_1}{\partial \theta}$$

with similar equations in $D_t e$ and $D_t\gamma$. From the same three equations we have

$$\frac{\partial c_1}{\partial \alpha}D_t\alpha + \frac{\partial c_1}{\partial e}D_te + \frac{\partial c_1}{\partial \gamma}D_t\gamma = \frac{\partial P_1}{\partial l}$$

$$\frac{\partial c_2}{\partial \alpha}D_t\alpha + \frac{\partial c_2}{\partial e}D_te + \frac{\partial c_2}{\partial \gamma}D_t\gamma = \frac{\partial P_1}{\partial \pi}$$

$$\frac{\partial c_3}{\partial \alpha}D_t\alpha + \frac{\partial c_3}{\partial e}D_te + \frac{\partial c_3}{\partial \gamma}D_t\gamma = \frac{\partial P_1}{\partial \theta}$$

It follows that if we solve these three equations for $D_t\alpha$, D_te, and $D_t\gamma$ the nine partial derivatives required will be the coefficients of the second members in the solution. Replacing the coefficients of the unknowns by their numerical values (25), we may reduce the solution to that of three numerical equations

$$
\begin{aligned}
0.494369X - .000777Y + .000671Z &= P \\
- .000435X - .052209Y - .000025Z &= Q \\
- .002033X + .000191Y - .179538Z &= R
\end{aligned}
$$

The solution of these equations so as to express X, Y, and Z as linear functions of P, Q, and R gives the following values of the factors which we seek. Along with these values is given for comparison the values found in *Action of Planets*, p. 196, where the numbers are the coefficients of $\frac{an}{m_2 m_3}$. The two determinations are completely independent, in that the earlier one is derived from the analytic expressions for the coördinates of the Moon, while these last have been obtained from Delaunay's expressions of the canonical elements $L\ G\ H$ in terms of a, e, γ.

$$
\begin{array}{lll}
\alpha_1 = +2.0228 & \text{Former value:} & +2.0225 \\
\alpha_2 = -0.0301 & & -0.0293 \\
\alpha_3 = +0.0075 & & +0.0075 \\
e_1 = -0.0168 & & -0.0169 \\
e_2 = -19.1534 & & -19.151 \\
e_3 = +0.0026 & & +0.0017 \\
\gamma_1 = -0.0229 & & -0.0233 \\
\gamma_2 = -0.0200 & & -0.0216 \\
\gamma_3 = -5.5700 & & -5.5704
\end{array}
\tag{26}
$$

The fundamental differential equations for the variations of the elements now become

$$
\begin{aligned}
a^2 n D_t \alpha &= \alpha_1 \frac{\partial P_1}{\partial l} + \alpha_2 \frac{\partial P_1}{\partial \pi} + \alpha_3 \frac{\partial P_1}{\partial \theta} \\
a^2 n D_t e &= e_1 \frac{\partial P_1}{\partial l} + e_2 \frac{\partial P_1}{\partial \pi} + e_3 \frac{\partial P_1}{\partial \theta} \\
a^2 n D_t \gamma &= \gamma_1 \frac{\partial P_1}{\partial l} + \gamma_2 \frac{\partial P_1}{\partial \pi} + \gamma_3 \frac{\partial P_1}{\partial \theta} \\
a^2 n D_t l_0 &= -\alpha_1 \frac{\partial P_1}{\partial \alpha} - e_1 \frac{\partial P_1}{\partial e} - \gamma_1 \frac{\partial P_1}{\partial \gamma} \\
a^2 n D_t \pi_0 &= -\alpha_2 \frac{\partial P_1}{\partial \alpha} - e_2 \frac{\partial P_1}{\partial e} - \gamma_2 \frac{\partial P_1}{\partial \gamma} \\
a^2 n D_t \theta_0 &= -\alpha_3 \frac{\partial P_1}{\partial \alpha} - e_3 \frac{\partial P_1}{\partial e} - \gamma_3 \frac{\partial P_1}{\partial \gamma}
\end{aligned}
\tag{27}
$$

$$[\text{Dim.} = ML^{-1}]$$

In order that we may, so far as possible, handle only pure numbers, with specifications of the units as concrete quantities, we shall substitute nt, the total motion of the Moon in mean longitude, and therefore a pure number, as the independent variable. The first numbers will then take the form $a^2n^2D_{nt}\alpha$, etc.

Since

$$a^2n^2 = \frac{\mu}{a}$$

the equations will now give

$$D_{nt}\alpha = \frac{a}{\mu}\left(\alpha_1\frac{\partial P_1}{\partial l} + \alpha_2\frac{\partial P_1}{\partial \pi} + \alpha_3\frac{\partial P_1}{\partial \theta}\right) \tag{28}$$

[Dim. = 0]

with five others formed in the same way from (27) which need not be written.

§ 15. *Elimination of t from the partial derivatives of l, π, and θ.*

An important remark at this point is that since P_1 is a function of l, π, and θ, the three quantities α, e, and γ enter into P_1 not only explicitly but implicitly through n, π, and θ, so that the complete differential variations of these functions are

$$\frac{dl}{dt} = \frac{dl_0}{dt} + n + t\frac{dn}{dt} \qquad \frac{d\pi}{dt} = \frac{d\pi_0}{dt} + \pi_1 + t\frac{d\pi_1}{dt} \qquad \frac{d\theta}{dt} = \frac{d\theta_0}{dt} + \theta_1 + t\frac{d\theta_1}{dt}$$

P_1 being a function of the six quantities

$$a,\ e,\ \gamma,\ l_0 + nt,\ \pi_0 + \pi_1 t,\ \theta_0 + \theta_1 t$$

its complete derivatives as to α, e, and γ are

$$\frac{\partial P_1}{\partial \alpha} = \left(\frac{\partial P_1}{\partial \alpha}\right) + t\left(\frac{\partial P_1}{\partial l}\frac{\partial n}{\partial \alpha} + \frac{\partial P_1}{\partial \pi}\frac{\partial \pi_1}{\partial \alpha} + \frac{\partial P_1}{\partial \theta}\frac{\partial \theta_1}{\partial \alpha}\right)$$

with similar expressions for $\partial P_1/\partial e$ and $\partial P_1/\partial \gamma$. Thus we have for any canonical element c

$$\frac{\partial P_1}{\partial c} = \left(\frac{\partial P_1}{\partial c}\right) + t\left(\frac{\partial P_1}{\partial l}\frac{\partial n}{\partial c} + \frac{\partial P_1}{\partial \pi}\frac{\partial \pi_1}{\partial c} + \frac{\partial P_1}{\partial \theta}\frac{\partial \theta_1}{\partial c}\right)$$

The complete derivatives of l, π, and θ are therefore

$$\frac{dl}{dt} = \frac{dl_0}{dt} + n + t\left(\frac{dn}{dt} - \frac{\partial P_1}{\partial l}\frac{\partial n}{\partial c_1} - \frac{\partial P_1}{\partial \pi}\frac{\partial \pi_1}{\partial c_1} - \frac{\partial P_1}{\partial \theta}\frac{\partial \theta_1}{\partial c_1}\right)$$

$$\frac{d\pi}{dt} = \frac{d\pi_0}{dt} + \pi_1 + t\left(\frac{d\pi_1}{dt} - \frac{\partial P_1}{\partial l}\frac{\partial n}{\partial c_2} - \frac{\partial P_1}{\partial \pi}\frac{\partial \pi_1}{\partial c_2} - \frac{\partial P_1}{\partial \theta}\frac{\partial \theta_1}{\partial c_2}\right)$$

$$\frac{d\theta}{dt} = \frac{d\theta_0}{dt} + \theta_1 + t\left(\frac{d\theta_1}{dt} - \frac{\partial P_1}{\partial l}\frac{\partial n}{\partial c_3} - \frac{\partial P_1}{\partial \pi}\frac{\partial \pi_1}{\partial c_3} - \frac{\partial P_1}{\partial \theta}\frac{\partial \theta_1}{\partial c_3}\right)$$

It is a fundamental theorem of the development of the planetary coördinates in periodic series that the terms of these equations containing t as a factor all vanish.* The values of l, π, and θ are therefore

$$l = l_0 + \int n\,dt \qquad \pi = \pi_0 + \int \pi_1 dt \qquad \theta = \theta_0 + \int \theta_1 dt \tag{29}$$

---

* A demonstration of this theorem in the most general case is found in the author's paper *On the General Integrals of Planetary Motion: Smithsonian Contributions to Knowledge*, 1874.

CHAPTER III.

DEFINITIVE FORM OF THE DIFFERENTIAL VARIATIONS OF THE ELEMENTS.

§ 16. The differential equations (27) in the form (28) are the fundamental ones of our problem, the integration of which is to be effected. This need be done only to terms of the first order as to the disturbing function. This amounts to saying that we regard the second members of the equation as known functions of the time, and that the required integration is to be performed by simple quadrature.

We begin by studying the general form of the function P_1. Besides ΔR, this function consists of two parts, one, R, arising from the direct action shown in § 7, and the other Ω_p arising from the indirect action. We have reduced both these parts to the general form

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz$$

The coefficients, A, B, etc., are functions of the heliocentric coördinates of two points: the centre of gravity G of the Earth and Moon, and that of the planet. They are, therefore, regarded as independent of the elements of the Moon's orbit. The variables x^2, y^2, etc., being functions of the geocentric coördinates of the Moon, are independent of the position of the planet, and contain, besides the six lunar elements proper, the major axis and eccentricity of the Earth's orbit around the Sun. The arguments on which the coefficients A, B, etc., depend are g_4 and g'. The coördinates x^2, y^2, etc., depend on the four arguments l, π, θ, and g'. It follows that the terms of P_1 depend on the five arguments

$$l, \ \pi, \ \theta, \ g', \ g_4$$

Although the two actions, the direct and indirect, admit of being treated together by combining the corresponding coefficients of x^2, y^2, etc., yet the coefficients are so different in their form and origin that it will be better to treat them separately.

§ 17. ***Reduction of the equations for the direct action.***

We begin with the development of R, as given by (13) and (14). Since

x^2, y^2, etc., each = a pur

A, B, C, etc., e

it follows that R may be developed in the form

$$R = \tfrac{3}{2} m_4 \frac{a^2}{a'^3} H \tag{30}$$

H being a pure number.

When the fundamental equations are taken in the form (28), and P_1 is replaced by R expressed in terms of H, the second members will all take the common constant numerical factor

$$\frac{3}{2} \frac{m_4}{\mu} \frac{a^3}{a'^3}$$

This factor may be simplified by the fundamental relations

$$a^3 n^2 = \mu \qquad a'^3 n'^2 = m' + \mu$$

where μ and m' are the respective masses of Earth + Moon and of the Sun.

Owing to the minuteness of μ relative to m' (1:330000 +) we may drop it from the quotient, thus obtaining

$$\frac{m'}{\mu} \frac{a^3}{a'^3} = \frac{n'^2}{n^2} = m^2$$

The factor thus reduces to the pure number

$$\frac{3}{2} \frac{m_4}{m'} m^2$$

The ratio $m_4 : m'$ is what is commonly taken as the numerical expression of the mass of the planet. We shall write

$$M = \frac{3}{2} \frac{m_4}{m'} m^2 = 0.008\,392\,86 \frac{m_4}{m'}$$

The numerical values of M for the four planets whose action is to be determined are as follows:

	$\frac{m'}{m^4}$	M
Venus	408 000	0''.004 242
Mars	3 093 500	0 .000 560
Jupiter	1047.35	1 .653
Saturn	3500	0 .4947

We have next to consider H and its derivatives. As this quantity has been above introduced we have

$$H = a'^3 A \frac{x^2}{a^2} + a'^3 B \frac{y^2}{a^2} + a'^3 C \frac{z^2}{a} + 2a'^3 D \frac{xy}{a} \tag{31}$$

The terms in E and F are omitted here, owing to their minuteness.

We have now to deal with two sets of factors:

1. The planetary factors, a'^3A, a'^3B, etc.
2. The lunar factors

$$\frac{x^2}{a^2}, \frac{y^2}{a^2}, \frac{z^2}{a^2}, \frac{2xy}{a^2}, \text{ etc.}$$

for which we use

$$\xi^2, \eta^2, \zeta^2, 2\xi\eta, \text{ respectively}$$

§ 18. *Notation of the Planetary Factors.* The development of these requires numerical processes which, owing to their length and their distinctive character, are given in Part II. We shall therefore assume this development to be effected, referring to Part II for the methods and numerical results. Considering the latter in their general form, we remark that these coefficients being of dimensions L^{-3}, if we compute their values, taking the Earth's mean distance as unity, the numbers obtained for the several coefficients A, B, etc., will readily be the values of a'^3A, a'^3B, etc. We shall therefore put

$$\begin{aligned} a'^3A &= \Sigma(A_c \cos N_4 + A_s \sin N_4) \\ a'^3B &= \Sigma(B_c \cos N_4 + B_s \sin N_4) \\ a'^3C &= \Sigma(C_c \cos N_4 + C_s \sin N_4) \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{aligned} \tag{33}$$

where each argument is of the general form

$$N_4 = kl_4 + k'g'$$

l_4 being the mean longitude of the planet, measured from a point which we shall take as that corresponding to the earth's perihelion.

§ 19. *Notation of the lunar factors.* We have shown in *Action*, Chapter II, how, from Delaunay's results, the squares and products of the Moon's coördinates may be developed in the general form

$$\frac{x^2}{a^2} = \xi^2 = \Sigma\kappa_1 \cos N \qquad \frac{y^2}{a^2} = \eta^2 = \Sigma\kappa_2 \cos N \qquad \frac{z^2}{a^2} = \zeta^2 = \Sigma\kappa_3 \cos N \tag{34}$$

$$2\xi\eta = \Sigma\kappa_4 \sin N \qquad 2\xi\zeta = \Sigma\kappa_5 \sin N \qquad 2\eta\zeta = \Sigma\kappa_6 \cos N$$

Here the κ are functions of a, e, γ, a', and e', and the arguments N may be expressed in the general form

$$N = il + i'\pi + i''\theta + jg'$$

These developments comprise all the quantities necessary to the formation of R and its derivatives.

§ 20. *Numerical form of the fundamental coefficients.* The condition

$$A + B + C = 0$$

enables us to reduce by one the number of terms in H, and at the same time to simplify the computation. We have the identity

$$A\xi^2 + B\eta^2 = \tfrac{1}{2}(A + B)(\xi^2 + \eta^2) + \tfrac{1}{2}(A - B)(\xi^2 - \eta^2)$$

Replacing $A + B$ by $-\ C$ there results

$$A\xi^2 + B\eta^2 + C\zeta^2 = \tfrac{1}{2}(A - B)(\xi^2 - \eta^2) - \tfrac{1}{2}C(\xi^2 + \eta^2 - 2\zeta^2)$$

Putting, for brevity,

$$K = \tfrac{1}{2}a'^3(A - B) \qquad C_1 = a'^3 C \qquad \rho^2 = \frac{r^2}{a^2} = \xi^2 + \eta^2 + \zeta^2 \qquad D_1 = a'^3 D$$

which will make K, C_1, and D_1 pure numbers, we shall have

$$H = K(\xi^2 - \eta^2) - \tfrac{1}{2}C_1(\rho^2 - 3\zeta^2) + 2D_1\xi\eta$$

The planetary factors, K, C_1, and D_1 are taken as developed in a double trigonometric series from the equations (33), by putting

$$K_c = \tfrac{1}{2}(A_c - B_c) \qquad K_s = \tfrac{1}{2}(A_s - B_s)$$

We shall then have for H the double trigonometric series

$$\begin{aligned} H = {} & \Sigma(K_c \cos N_4 + K_s \sin N_4)(\kappa_1 - \kappa_2)\cos N \\ & - \Sigma(\tfrac{1}{2}C_c \cos N_4 + \tfrac{1}{2}C_s \sin N_4)(\kappa_1 + \kappa_2 - 2\kappa_3)\cos N \\ & + \Sigma(D_c \cos N_4 + D_s \sin N_4)\kappa_4 \sin N \end{aligned} \tag{35}$$

Introducing, for brevity,

$$p = \tfrac{1}{2}(\kappa_1 - \kappa_2) \qquad q = \tfrac{1}{2}(\kappa_1 + \kappa_2) - \kappa_3 \tag{36}$$

the terms of the lunar factors will be expressed by

$$(\xi^2 - \eta^2) = 2p \cos N \qquad \rho^2 - 3\zeta^2 = 2q \cos N \qquad 2\xi\eta = \kappa_4 \sin N.$$

Every combination of a planetary argument N_4 with a lunar argument N will give rise to a set of terms in H of the form

$$H = h_c \cos(N + N_4) + h_s \sin(N + N_4) + h_c' \cos(N - N_4) + h_s' \sin(N - N_4) \tag{37}$$

where

$$\begin{aligned} h_c &= K_c p - \tfrac{1}{2}C_c q - \tfrac{1}{2}D_s\kappa_4 \qquad & h_c' &= -K_c p - \tfrac{1}{2}C_c q + \tfrac{1}{2}D_s\kappa_4 \\ h_s &= K_s p - \tfrac{1}{2}C_s q + \tfrac{1}{2}D_c\kappa_4 \qquad & h_s' &= -K_s p + \tfrac{1}{2}C_s q + \tfrac{1}{2}D_c\kappa_4 \end{aligned} \tag{38}$$

The partial derivatives of H as to α, e, and γ are to be found from

$$
\begin{aligned}
Dh_c &= K_c Dp - \tfrac{1}{2} C_c Dq - \tfrac{1}{2} D_s D\kappa_4 \\
\frac{\partial h_c}{\partial e} &= K_c \frac{\partial p}{\partial e} - \tfrac{1}{2} C_c \frac{\partial q}{\partial e} - \tfrac{1}{2} D_s \frac{\partial \kappa_4}{\partial e} \\
\frac{\partial h_c}{\partial \gamma} &= K_c \frac{\partial p}{\partial \gamma} - \tfrac{1}{2} C_c \frac{\partial q}{\partial \gamma} - \tfrac{1}{2} D_s \frac{\partial \kappa_4}{\partial \gamma}
\end{aligned}
\tag{39}
$$

with three other sets formed by replacing p, q, and κ_4 in (38) by their partial derivatives. These derivatives of h_c, h_s, h_c', and h_s' being substituted in (37) give the required partial derivatives of H. In forming the derivatives as to l, π, and θ we note that these quantities enter only through the arguments N, in which they have the respective coefficients

$$i;\quad i';\quad i''$$

Their formation is therefore a simple algebraic process after H is developed.

The elements e and γ also enter R only through H. But a appears both in H, which is a function of m, and in the factor a^2/a'^3. We therefore have from (30)

$$\frac{\partial R}{\partial \alpha} = \tfrac{3}{2} m_4 \frac{a^2}{a'^3}\left(2H + \frac{\partial H}{\partial \alpha}\right)$$

For consistency in form and notation we shall put

$$D'H = 2H + \frac{\partial H}{\partial \alpha} \tag{40}$$

It may be remarked that the formation of $D'H$ may be effected by the general operation indicated in (23), by supposing H developed in powers of m and putting

$$M = a^2 H$$

so that

$$i = 2$$

We then have

$$H = M_0 + M_1 m + M_2 m^2 + \cdots$$

and

$$\frac{\partial H}{\partial \alpha} = \tfrac{3}{2} M_1 m + \tfrac{6}{2} M_2 m^2 + \cdots$$

The sum of this $+2H$ gives $D'H$ as above expressed. In forming this sum we need not use the analytic development of $2H$, which is necessary to form $\partial H/\partial \alpha$, but may use the numerical development when it is more accurate.

The partial derivatives of R as to α, e, and γ are

$$\frac{\partial R}{\partial \alpha} = \tfrac{3}{2} m_4 \frac{a^2}{a'^3} D'H \qquad \frac{\partial R}{\partial e} = \tfrac{3}{2} m_4 \frac{a^2}{a'^3}\frac{\partial H}{\partial e} \qquad \frac{\partial R}{\partial \gamma} = \tfrac{3}{2} m_4 \frac{a^2}{a'^3}\frac{\partial H}{\partial \gamma}$$

§ 21. The fundamental equations in the form (28) for the direct action now become

$$D_{nt}\alpha = M\left(\alpha_1\frac{\partial H}{\partial l} + \alpha_2\frac{\partial H}{\partial \pi} + \alpha_3\frac{\partial H}{\partial \theta}\right)$$

$$D_{nt}e = M\left(e_1\frac{\partial H}{\partial l} + e_2\frac{\partial H}{\partial \pi} + e_3\frac{\partial H}{\partial \theta}\right) \qquad (41)$$

$$D_{nt}\gamma = M\left(\gamma_1\frac{\partial H}{\partial l} + \gamma_2\frac{\partial H}{\partial \pi} + \gamma_3\frac{\partial H}{\partial \theta}\right)$$

$$-D_{nt}l_0 = M\left(\alpha_1 D'H + e_1\frac{\partial H}{\partial e} + \gamma_1\frac{\partial H}{\partial \gamma}\right)$$

$$-D_{nt}\pi_0 = M\left(\alpha_2 D'H + e_2\frac{\partial H}{\partial e} + \gamma_2\frac{\partial H}{\partial \gamma}\right) \qquad (42)$$

$$-D_{nt}\theta_0 = M\left(\alpha_3 D'H + e_3\frac{\partial H}{\partial e} + \gamma_3\frac{\partial H}{\partial \gamma}\right)$$

§ 22. We have next to show how the second members of these equations may be most readily reduced to numbers. There being a certain number of lunar arguments N and also a certain number of planetary arguments N_4, it will conduce to simplicity to carry forward the quantities depending on the argument of each class as far as possible before making the combination.

Each lunar argument being of the general form

$$N = il + i'\pi + i''\theta + jg'$$

and each planetary one of the form

$$N_4 = k'g' + kl_4$$

it follows that by putting G for the general value of the combined final argument, $N \pm N_4$,

$$G = il + i'\pi + i''\theta + (j \pm k')g' \pm kl_4$$

the general form (37) of H may be written

$$H = \Sigma(h_c \cos G + h_s \sin G)$$

The derivatives of H as to l, π, and θ are

$$\frac{\partial H}{\partial l} = \Sigma(-ih_c \sin G + ih_s \cos G) \qquad \frac{\partial H}{\partial \pi} = \Sigma(-i'h_c \sin G + i'h_s \cos G)$$

$$\frac{\partial H}{\partial \theta} = \Sigma(-i''h_c \sin G + i''h_s \cos G)$$

Substituting these values in (41) and putting

$$\mathrm{a} = i\alpha_1 + i'\alpha_2 + i''\alpha_3 \qquad \mathrm{e} = ie_1 + i'e_2 + i''e_3 \qquad \mathrm{g} = i\gamma_1 + i'\gamma_2 + i''\gamma_3 \qquad (43)$$

the equations (41) become

$$\begin{aligned} D_{nt}\alpha &= M(\mathrm{a}h_s \cos G - \mathrm{a}h_c \sin G) \\ D_{nt}e &= M(\mathrm{e}h_s \cos G - \mathrm{e}h_c \sin G) \\ D_{nt}\gamma &= M(\mathrm{g}h_s \cos G - \mathrm{g}h_c \sin G) \end{aligned} \tag{44}$$

Every combination of a lunar argument N with a planetary argument N_4 gives rise in each derivative of an element to four terms, which we shall express in the form

$$D_{nt}\alpha = h_{\alpha,c} \cos(N + N_4) + h_{\alpha,s} \sin(N + N_4) + h_{\alpha,c}' \cos(N - N_4) + h_{\alpha,s}' \sin(N - N_4) \tag{45}$$

Replacing h_s and h_c in (44) by their values (38) we have for each combination

$$\begin{aligned} h_{\alpha,c} &= + MK_s\mathrm{a}p - \tfrac{1}{2}MC_s\mathrm{a}q + \tfrac{1}{2}MD_c\mathrm{a}\kappa_4 \\ h_{\alpha,c}' &= - MK_s\mathrm{a}p + \tfrac{1}{2}MC_s\mathrm{a}q + \tfrac{1}{2}MD_c\mathrm{a}\kappa_4 \\ h_{\alpha,s} &= - MK_c\mathrm{a}p + \tfrac{1}{2}MC_c\mathrm{a}q + \tfrac{1}{2}MD_s\mathrm{a}\kappa_4 \\ h_{\alpha,s}' &= - MK_s\mathrm{a}p + \tfrac{1}{2}MC_s\mathrm{a}q - \tfrac{1}{2}MD_s\mathrm{a}\kappa_4 \end{aligned} \tag{46}$$

$$\begin{aligned} h_{e,c} &= + MK_s\mathrm{e}p - \tfrac{1}{2}MC_s\mathrm{e}q + \tfrac{1}{2}MD_c\mathrm{e}\kappa_4 \\ h_{e,c}' &= - MK_s\mathrm{e}p + \tfrac{1}{2}MC_s\mathrm{e}q + \tfrac{1}{2}MD_c\mathrm{e}\kappa_4 \\ h_{e,s} &= - MK_c\mathrm{e}p + \tfrac{1}{2}MC_c\mathrm{e}q + \tfrac{1}{2}MD_s\mathrm{e}\kappa_4 \\ h_{e,s}' &= - MK_s\mathrm{e}p + \tfrac{1}{2}MC_s\mathrm{e}q - \tfrac{1}{2}MD_s\mathrm{e}\kappa_4 \end{aligned} \tag{47}$$

$$\begin{aligned} h_{\gamma,c} &= + MK_s\mathrm{g}p - \tfrac{1}{2}MC_s\mathrm{g}q + \tfrac{1}{2}MD_c\mathrm{g}\kappa_4 \\ h_{\gamma,c}' &= - MK_s\mathrm{g}p + \tfrac{1}{2}MC_s\mathrm{g}q + \tfrac{1}{2}MD_c\mathrm{g}\kappa_4 \\ h_{\gamma,s} &= - MK_c\mathrm{g}p + \tfrac{1}{2}MC_c\mathrm{g}q + \tfrac{1}{2}MD_s\mathrm{g}\kappa_4 \\ h_{\gamma,s}' &= - MK_s\mathrm{g}p + \tfrac{1}{2}MC_s\mathrm{g}q - \tfrac{1}{2}MD_s\mathrm{g}\kappa_4 \end{aligned} \tag{48}$$

§ 23. We now reduce in a similar way the group (42). We have for each argument,

$$D'H = D'h_c \cos G + D'h_s \sin G \qquad \frac{\partial H}{\partial e} = \frac{\partial h_c}{\partial e} \cos G + \frac{\partial h_s}{\partial e} \sin G \qquad \frac{\partial H}{\partial \gamma} = \frac{\partial h_c}{\partial \gamma} \cos G + \frac{\partial h_s}{\partial \gamma} \sin G$$

Replacing h_c and h_s by their values (38) and substituting the resulting partial derivatives in (42) we have results which we may write in the form

$$\begin{aligned} -D_{nt}l_0 &= h_{l,c} \cos(N + N_4) + h_{l,s} \sin(N + N_4) + h_{l,c}' \cos(N - N_4) + h_{l,s}' \sin(N - N_4) \\ -D_{nt}\pi_0 &= h_{\pi,c} \cos(N + N_4) + h_{\pi,s} \sin(N + N_4) + h_{\pi,c}' \cos(N - N_4) + h_{\pi,s}' \sin(N - N_4) \\ -D_{nt}\theta_0 &= h_{\theta,c} \cos(N + N_4) + h_{\theta,s} \sin(N + N_4) + h_{\theta,c}' \cos(N - N_4) + h_{\theta,s}' \sin(N - N_4) \end{aligned} \tag{49}$$

where the values of the coefficients are found by the following computation. For each lunar argument we form

$$
\begin{array}{lll}
L' = \alpha_1 D'p + e_1 \dfrac{\partial p}{\partial e} + \gamma_1 \dfrac{\partial p}{\partial \gamma} & L'' = \alpha_1 D'q + e_1 \dfrac{\partial q}{\partial e} + \gamma_1 \dfrac{\partial q}{\partial \gamma} & L_4 = \alpha_1 D'\kappa_4 + e_1 \dfrac{\partial \kappa_4}{\partial e} + \gamma_1 \dfrac{\partial \kappa_4}{\partial \gamma} \\[2ex]
P' = \alpha_2 D'p + e_2 \dfrac{\partial p}{\partial e} + \gamma_2 \dfrac{\partial p}{\partial \gamma} & P'' = \alpha_2 D'q + e_2 \dfrac{\partial q}{\partial e} + \gamma_2 \dfrac{\partial q}{\partial \gamma} & P_4 = \alpha_2 D'\kappa_4 + e_2 \dfrac{\partial \kappa_4}{\partial e} + \gamma_2 \dfrac{\partial \kappa_4}{\partial \gamma} \\[2ex]
R' = \alpha_3 D'p + e_3 \dfrac{\partial p}{\partial e} + \gamma_3 \dfrac{\partial p}{\partial \gamma} & R'' = \alpha_3 D'q + e_3 \dfrac{\partial q}{\partial e} + \gamma_3 \dfrac{\partial q}{\partial \gamma} & R_4 = \alpha_3 D'\kappa_4 + e_3 \dfrac{\partial \kappa_4}{\partial e} + \gamma_3 \dfrac{\partial \kappa_4}{\partial \gamma}
\end{array}
\qquad (50)
$$

Then for each pair of arguments

$$
\begin{array}{ll}
h_{l,c} = MK_c L' - \tfrac{1}{2} MC_s L'' - \tfrac{1}{2} MD_s L_4 & h_{l,c}' = + MK_c L' - \tfrac{1}{2} MC_c L'' + \tfrac{1}{2} MD_s L_4 \\[1ex]
h_{l,s} = MK_s L' - \tfrac{1}{2} MC_s L'' + \tfrac{1}{2} MD_c L_4 & h_{l,s}' = - MK_s L' + \tfrac{1}{2} MC_s L'' + \tfrac{1}{2} MD_c L_4 \\[1ex]
h_{\pi,c} = MK_c P' - \tfrac{1}{2} MC_c P'' - \tfrac{1}{2} MD_s P_4 & h_{\pi,c}' = + MK_c P' - \tfrac{1}{2} MC_c P'' + \tfrac{1}{2} MD_s P_4 \\
h_{\pi,s} = MK_s P' - \tfrac{1}{2} MC_s P'' + \tfrac{1}{2} MD_c P_4 & h_{\pi,s}' = - MK_s P' + \tfrac{1}{2} MC_s P'' + \tfrac{1}{2} MD_c P_4 \\[1ex]
h_{\theta,c} = MK_c R' - \tfrac{1}{2} MC_c R'' - \tfrac{1}{2} MD_s R_4 & h_{\theta,c}' = + MK_c R' - \tfrac{1}{2} MC_c R'' + \tfrac{1}{2} MD_s R_4 \\[1ex]
h_{\theta,s} = MK_s R' - \tfrac{1}{2} MC_s R'' + \tfrac{1}{2} MD_c R_4 & h_{\theta,s}' = - MK_s R' + \tfrac{1}{2} MC_s R'' + \tfrac{1}{2} MD_c R_4
\end{array}
\qquad (51)
$$

§ 24. *Development of the indirect action.*

The fundamental equations for the indirect action are found from (28) by replacing P_1 by the function Ω_p defined in (12). We first replace the coefficients δT by the following:

$$A' = a'^3 \delta T_1 \qquad B' = a'^3 \delta T_2, \text{ etc.}$$

Taking, as we do throughout this work, the mean Sun as the origin of longitudes, the true longitude, v', will be replaced by the Sun's equation of the centre $\equiv E$. We also put

$$r_1 = \frac{r'}{a'}$$

With these substitutions the equations (11) will be replaced by others which may be written thus: Put

$$
\begin{aligned}
G &= \tfrac{3}{2} r_1^{-3} \sin 2E\, \delta v' + \tfrac{9}{4} r_1^{-3} \cos 2E\, \delta\rho' \\
J &= \tfrac{3}{4} r_1^{-3} \delta\rho' \\
I &= \tfrac{3}{2} r_1^{-3} \cos 2E\, \delta v' - \tfrac{9}{4} r_1^{-3} \sin 2E\, \delta\rho'
\end{aligned}
\qquad (52)
$$

Then

$$
\begin{array}{cccc}
A' = -G - J & B' = G - J & C' = 2J & D' = I \\
\multicolumn{2}{c}{E' = \tfrac{3}{2} r_1^{-3} \cos E \sin \beta'} & \multicolumn{2}{c}{F' = \tfrac{3}{2} r_1^{-3} \sin E \sin \beta'}
\end{array}
\qquad (53)
$$

These substitutions lead to the replacement of expression (12) by

$$\Omega_p = \frac{m' a^2}{a'^3} H' \tag{54}$$

where

$$H' = A'\xi^2 + B'\eta^2 + C'\xi^2 + 2D'\xi\eta + \cdots \tag{55}$$

This function H', a pure number in dimensions, will hereafter be used as a fundamental quantity instead of Ω^p.

By replacing P_1 by this value of Ω_p in (27) the second members in the form (28) take the common factor

$$\frac{m' a^3}{\mu a'^3} = m^2$$

and the differential variations of the elements become

$$\left.\begin{aligned}
D_w a &= m^2\left(\alpha_1\frac{\partial H'}{\partial l} + \alpha_2\frac{\partial H'}{\partial \pi} + \alpha_3\frac{\partial H'}{\partial \theta}\right)\\
D_w e &= m^2\left(e_1\frac{\partial H'}{\partial l} + e_2\frac{\partial H'}{\partial \pi} + e_3\frac{\partial H'}{\partial \theta}\right)\\
D_w \gamma &= m^2\left(\gamma_1\frac{\partial H'}{\partial l} + \gamma_2\frac{\partial H'}{\partial \pi} + \gamma_3\frac{\partial H'}{\partial \theta}\right)
\end{aligned}\right\} \tag{56}$$

$$\left.\begin{aligned}
-D_w l_0 &= m^2\left(\alpha_1 D'H' + e_1\frac{\partial H'}{\partial e} + \gamma_1\frac{\partial H'}{\partial \gamma}\right)\\
-D_w \pi_0 &= m^2\left(\alpha_2 D'H' + e_2\frac{\partial H'}{\partial e} + \gamma_2\frac{\partial H'}{\partial \gamma}\right)\\
-D_w \theta_0 &= m^2\left(\alpha_3 D'H' + e_3\frac{\partial H'}{\partial e} + \gamma_3\frac{\partial H'}{\partial \gamma}\right)
\end{aligned}\right\} \tag{57}$$

We have next to develop the values (52) of G, J, and I in terms of the mean anomaly g'. This may be done by means of Cayley's tables in the *Memoirs of the Royal Astronomical Society*, Vol. XXIX, or the development given by Leverrier in *Annales de l'Observatoire de Paris*, Vol. I. Dropping unnecessary terms and powers of e' we have

$$\left.\begin{aligned}
r_1^{-3}\cos 2E &= 1 - \tfrac{5}{2}e'^2 + (3e' - \tfrac{61}{8}e'^3)\cos g' + \tfrac{17}{2}e'^2\cos 2g'\\
r_1^{-3}\sin 2E &= (4e' - \tfrac{21}{4}e'^3)\sin g' + \tfrac{17}{2}e'^2\sin 2g'\\
r_1^{-3} &= 1 + \tfrac{3}{2}e'^2 + (3e' + \tfrac{27}{8}e'^3)\cos g' + \tfrac{9}{2}e'^2\cos 2g\\
r_1^{-3}\cos E &= 1 + 3e'\cos g'\\
r_1^{-3}\sin E &= 2e'\sin g'
\end{aligned}\right\} \tag{58}$$

The expressions for G, J, and I thus become

$$\begin{aligned} G &= \{(6e' - \tfrac{93}{8}e'^3)\sin g' + \tfrac{51}{4}e'^2 \sin 2g'\}\delta v' \\ &\quad + \{\tfrac{9}{4} - \tfrac{45}{8}e'^2 + (\tfrac{27}{4}e' - \tfrac{549}{32}e'^3)\cos g' + \tfrac{153}{8}e'^2 \cos 2g'\}\delta\rho' \\ J &= -\{\tfrac{3}{4} + \tfrac{9}{8}e'^2 + (\tfrac{9}{4}e' + \tfrac{81}{32}e'^3)\cos g' + \tfrac{27}{8}e'^2 \cos 2g'\}\delta\rho' \\ I &= \{\tfrac{3}{2} - \tfrac{15}{4}e'^2 + (\tfrac{9}{2}e' - \tfrac{183}{16}e'^3)\cos g' + \tfrac{51}{4}e'^2 \cos 2g'\}\delta v' \\ &\quad - \{(9e' - \tfrac{279}{16}e'^3)\sin g' + \tfrac{153}{8}e'^2 \sin 2g'\}\delta\rho' \end{aligned} \tag{59}$$

In reducing these expressions to numbers I take, with Delaunay and Brown, the value of e' for 1850

$$e' = .016\,771$$

With this datum the expressions for G, J, etc., become

$$\begin{aligned} G &= (0.10058 \sin g' + 0.00359 \sin 2g')\delta v' \\ &\quad + (2.24842 + 0.11313 \cos g' + 0.00538 \cos 2g')\delta\rho' \\ J &= (0.75032 + 0.03775 \cos g' + 0.00095 \cos 2g')\delta\rho' \\ I &= (1.49895 + 0.07542 \cos g' + 0.00359 \cos 2g')\delta v' \\ &\quad - (0.15087 \sin g' + 0.00538 \sin 2g')\delta\rho' \end{aligned} \tag{60}$$

§ 25. *Abbreviated coefficients for the indirect action.* Since

$$A' + B' + C' = 0$$

we have, as in the direct action,

$$H' = \tfrac{1}{2}(A' - B')(\xi^2 - \eta^2) - \tfrac{1}{2}C'(\xi^2 + \eta^2 - 2\zeta^2) + 2D'\xi\eta$$

Replacing A', B', and C' by their values (53)

$$H' = -G(\xi^2 - \eta^2) - J(\xi^2 + \eta^2 - 2\zeta^2) + 2I\xi\eta + 2E'\xi\zeta + 2F'\eta\zeta$$

As the last two terms of H' are important only in some exceptional cases, we postpone their development to Part IV.

With the notation of (36), we have for each lunar argument

$$H' = (-2Gp \cos N - 2Jq \cos N + I\kappa_4 \sin N) \tag{61}$$

The planetary factors, G, J, and I are to be developed in a periodic series of the same form as that for A, B, and C, so that, for each planetary argument N_4 we shall have

$$G = G_c \cos N_4 + G_s \sin N_4 \qquad J = J_c \cos N_4 + J_s \sin N_4 \qquad I = I_c \cos N_4 + I_s \sin N_4 \tag{62}$$

With these values we shall have H' developed in a double series in which for each pair of arguments N and N_4, H' will have the four terms

$$H' = h_c \cos (N + N_4) + h_c' \cos (N - N_4) + h_s \sin (N + N_4) + h_s' \sin (N - N_4) \qquad (63)$$

where

$$h_c = - G_c p - J_c q - \tfrac{1}{2} I_s \kappa_4 \qquad h_c' = - G_c p - J_c q + \tfrac{1}{2} I_s \kappa_4$$

$$h_s = - G_s p - J_s q + \tfrac{1}{2} I_c \kappa_4 \qquad h_s' = G_s p + J_s q + \tfrac{1}{2} I_c \kappa_4$$

Expressing the differential variations of the elements in the same form as before we shall find

$$\left.\begin{aligned} h_{\alpha,c} &= m^2(- G_s \mathrm{a} p - J_s \mathrm{a} q + \tfrac{1}{2} I_c \mathrm{a} \kappa_4) \qquad & h_{\alpha,c}' &= m^2(G_s \mathrm{a} p + J_s \mathrm{a} q + \tfrac{1}{2} I_c \mathrm{a} \kappa_4) \\ h_{\alpha,s} &= m^2(G_c \mathrm{a} p + J_c \mathrm{a} q + \tfrac{1}{2} I_s \mathrm{a} \kappa_4) \qquad & h_{\alpha,s}' &= m^2(G_c \mathrm{a} p + J_c \mathrm{a} q - \tfrac{1}{2} I_s \mathrm{a} \kappa_4) \end{aligned}\right\} \quad (64)$$

with two other sets of equations found by replacing α and a by e and e for the set in e, and by γ and g for the set in y. Also,

$$\left.\begin{aligned} h_{l,c} &= m^2(- G_s L' - J_s L'' - \tfrac{1}{2} I_c L_4) \qquad & h_{l,c}' &= m^2(- G_c L' - J_c L'' + \tfrac{1}{2} I_s L_4) \\ h_{l,s} &= m^2(- G_s L' - J_s L'' + \tfrac{1}{2} I_c L_4) \qquad & h_{l,s}' &= m^2(G_s L' + J_s L'' + \tfrac{1}{2} I_c L_4) \end{aligned}\right\} \quad (65)$$

with two other sets formed by replacing l and L by π and P, for the set in π, and by θ and R for the set in θ.

Comparing these with the corresponding coefficients (51) for the direct action we see that the equations for the indirect action may be formed from those of the direct action by replacing

$$K,\ \tfrac{1}{2}C \text{ and } D \text{ by } - G,\ J, \text{ and } I; \text{ and also } M \text{ by } m^2$$

It also follows that the two actions may be combined by replacing in the expressions for the coefficients h, given in (46), (47), (48) and (51),

$$MK \text{ by } MK - m^2 G; \qquad \tfrac{1}{2}MC \text{ by } \tfrac{1}{2}MC + m^2 J; \qquad MD \text{ by } MD + m^2 I \quad (66)$$

We shall make this combination to save labor in the formation of the products, but shall give the separate parts of the coefficients, so that the parts of each term due to the respective actions may be readily found.

§ 26. *Integration of the equations.* The integration is effected by multiplying each coefficient by the quotient of the mean motion of the Moon by the motion of the argument itself, which factor is

$$\nu = \frac{n}{in + i'\pi_1 + i''\theta_1 \pm (j + k')n' \pm kn_4} \qquad (67)$$

The reciprocal of this factor, which we may use as a divisor, is

$$\frac{1}{\nu} = i + i'\frac{\pi_1}{n} + i''\frac{\theta_1}{n} \pm (j + k')\frac{n'}{n} \pm k\frac{n_4}{n}$$

a form most convenient for numerical computation.

We shall thus have for the perturbations of the elements corresponding to each pair of lunar and planetary arguments

$$\begin{aligned} \delta\alpha &= \nu h_{\alpha,c} \sin(N \pm N_4) - \nu h_{\alpha,s} \cos(N \pm N_4) \\ \delta e &= \nu h_{e,c} \sin(N \pm N_4) - \nu h_{e,s} \cos(N \pm N_4) \\ \delta\gamma &= \nu h_{\gamma,c} \sin(N \pm N_4) - \nu h_{\gamma,s} \cos(N \pm N_4) \end{aligned} \tag{68}$$

$$\begin{aligned} \delta l_0 &= -\nu h_{l,c} \sin(N \pm N_4) + \nu h_{l,s} \cos(N \pm N_4) \\ \delta\pi_0 &= -\nu h_{\pi,c} \sin(N \pm N_4) + \nu h_{\pi,s} \cos(N \pm N_4) \\ \delta\theta_0 &= -\nu h_{\theta,c} \sin(N \pm N_4) + \nu h_{\theta,s} \cos(N \pm N_4) \end{aligned} \tag{69}$$

Practically we use the perturbation of n, the mean motion, instead of α. From the relation of § 12, β, we have

$$D_t n = -\tfrac{3}{2} n D_t \alpha$$

Thus the first equation (68) is replaced by

$$\delta n = -\tfrac{3}{2}\nu n h_{\alpha,c} \sin(N \pm N_4) + \tfrac{3}{2}\nu n h_{\alpha,s} \cos(N \pm N_4) \tag{70}$$

§ 27. We pass next to the inequalities of the actual mean longitude, l, and of the perigee and node, π and θ. Taking the equations (29) for these quantities

$$l = l_0 + \int n\,dt \qquad \pi = \pi_0 + \int \pi_1 dt \qquad \theta = \theta_0 + \int \theta_1 dt$$

the complete expressions are

$$\delta l = \delta l_0 + \int \delta n\,dt \qquad \delta\pi = \delta\pi_0 + \int \delta\pi_1 dt \qquad \delta\theta = \delta\theta_0 + \int \delta\theta_1 dt \tag{71}$$

The motions n, π_1, and θ_1 are functions of the elements a (or α), e, and γ. n is given by the relation $a^3 n^2 = \mu$, while π_1 and θ_1 have been developed by Delaunay, whose results are found in *Comptes Rendus*, Vol. LXXIV, 1872, I, and are reproduced in part in *Action*, p. 190.

$$\delta\pi_1 = \frac{\partial\pi_1}{\partial n}\delta n + \frac{\partial\pi_1}{\partial e}\delta e + \frac{\partial\pi_1}{\partial\gamma}\delta\gamma \qquad \delta\theta_1 = \frac{\partial\theta_1}{\partial n}\delta n + \frac{\partial\theta_1}{\partial e}\delta e + \frac{\partial\theta_1}{\partial\gamma}\delta\gamma$$

From (70) and (71) we thus have, in the variation of l, the terms

$$D_{nt}\delta l = -\tfrac{3}{2}\nu h_{\alpha,c} \sin(N \pm N_4) + \tfrac{3}{2}\nu h_{\alpha,s} \cos(N \pm N_4) \tag{72}$$

arising from the variation of n. Integrating and including the value of δl_0 we shall have for the complete perturbation of the mean longitude

$$\delta l = l_c \cos(N \pm N_4) + l_s \sin(N \pm N_4)$$

where

$$l_c = \tfrac{3}{2}\nu^2 h_{a,c} + \nu h_{l,s} = \nu(\tfrac{3}{2}\nu h_{a,c} + h_{l,s}) \qquad l_s = \tfrac{3}{2}\nu^2 h_{a,s} - \nu h_{l,c} = \nu(\tfrac{3}{2}\nu h_{a,s} - h_{l,c}) \tag{73}$$

From the Delaunay developments in powers of m are found

$$\begin{aligned} \frac{\partial\pi_1}{\partial n} &= -.01480 & \frac{\partial\pi_1}{\partial e} &= -.001042n & \frac{\partial\pi_1}{\partial\gamma} &= -.004331n \\ \frac{\partial\theta_1}{\partial n} &= +.00377 & \frac{\partial\theta_1}{\partial e} &= -.001292n & \frac{\partial\theta_1}{\partial\gamma} &= +.000665n \end{aligned} \tag{74}$$

Substituting these values and the values (68) and (70) we find that by putting

$$\begin{aligned} \pi_{1,c} &= .02220h_{a,c} - .00104h_{e,c} - .00433h_{\gamma,c} & \pi_{1,s} &= .02220h_{a,s} - .00104h_{e,s} - .00433h_{\gamma,s} \\ \theta_{1,c} &= -.00566h_{a,c} - .00129h_{e,c} + .00066h_{\gamma,c} & \theta_{1,s} &= -.00566h_{a,s} - .00129h_{e,s} + .00066h_{\gamma,s} \end{aligned} \tag{75}$$

we shall have

$$\delta\pi_1 = \nu n\{\pi_{1,c} \sin(N \pm N_4) - \pi_{1,s} \cos(N \pm N_4)\}$$

$$\delta\theta_1 = \nu n\{\theta_{1,c} \sin(N \pm N_4) - \theta_{1,s} \cos(N \pm N_4)\}$$

Then by integrating we have the terms

$$\begin{aligned} \delta\pi &= -\nu^2\pi_{1,c} \cos(N \pm N_4) - \nu^2\pi_{1,s} \sin(N \pm N_4) \\ \delta\theta &= -\nu^2\theta_{1,c} \cos(N \pm N_4) - \nu^2\theta_{1,s} \sin(N \pm N_4) \end{aligned} \tag{76}$$

Adding the values (69) we have the complete periodic perturbations of π and θ expressed in the form

$$\delta\pi = \pi_c \cos(N \pm N_4) + \pi_s \sin(N \pm N_4) \qquad \delta\theta = \theta_c \cos(N \pm N_4) + \theta_s \sin(N \pm N_4)$$

where

$$\begin{aligned} \pi_c &= \nu h_{\pi,s} - \nu^2\pi_{1,c} = \nu(h_{\pi,s} - \nu\pi_{1,c}) & \pi_s &= -\nu h_{\pi,c} - \nu^2\pi_{1,s} = -\nu(h_{\pi,c} + \nu\pi_{1,s}) \\ \theta_c &= \nu h_{\theta,s} - \nu^2\theta_{1,c} = \nu(h_{\theta,s} - \nu\theta_{1,c}) & \theta_s &= -\nu h_{\theta,c} - \nu^2\theta_{1,s} = -\nu(h_{\theta,c} + \nu\theta_{1,s}) \end{aligned} \tag{77}$$

§ 28. *Treatment of the non-periodic terms in R.*

In the preceding integration we have supposed all the arguments to be of the form $G_0 + \mathrm{N}t$. We have now to consider the special case in which N vanishes. In the case of the direct action this occurs when, in the pair of arguments which form G,

$$i = i' = i' = k = 0 \qquad j \pm k' = 0$$

We shall then have in H a term, $H = \Sigma h_c$, which we shall call h_c simply. It will be affected by a minute secular variation which we need not consider at present.

In the case of the indirect action we note that the coefficients of $\delta v'$, $\delta\rho'$, and $\delta\beta'$, as found in (59), are developed in the general form $\Sigma k_i \sin ig'$ or $k_i \cos ig'$, in which k_0, being a function of the eccentricity of the Earth's orbit, is a function of the time. The coefficients of all the terms arising from the indirect action are therefore affected by a secular variation.

The perturbations $\delta v'$ and $\delta\rho'$ contain terms independent of the mean longitude of the disturbing planet, which may be treated separately, namely:

(1) A constant term in $\delta\rho'$.
(2) Terms of the form $c \, {}^{\sin}_{\cos} \, ig'$ in $\delta v'$ and $\delta\rho'$.
(3) The secular variation of e' and of ρ'.

Omitting for the present the powers of t above the first, we shall have in $\delta v'$ and $\delta\rho'$ terms of the general form

$$(c + c't) \, {}^{\sin}_{\cos} \, ig'$$

The product of these into (59) gives rise to terms of G, J, and I of the same form. When we form the products of these terms by ξ^2, η^2, etc., we shall have in H' terms of the form

$$h + h'nt + \cdots$$

Substituting the derivatives of the non-periodic direct term in (41) and (42), and of the indirect term in (56) and (57), omitting terms in t, and putting for brevity

$$P_0 = Mh_0 + m^2 h$$

we find

$$D_{nt}\alpha = D_{nt}e = D_{nt}\gamma = 0$$

$$\left.\begin{aligned} -D_{nt}l_0 &= \alpha_1 D'P_0 + e_1 \frac{\partial P_0}{\partial e} + \gamma_1 \frac{\partial P_0}{\partial \gamma} \equiv h_0 \\ -D_{nt}\pi_0 &= \alpha_2 D'P_0 + e_2 \frac{\partial P_0}{\partial e} + \gamma_2 \frac{\partial P_0}{\partial \gamma} \equiv h_0' \\ -D_{nt}\theta_0 &= \alpha_3 D'P_0 + e_3 \frac{\partial P_0}{\partial e} + \gamma_3 \frac{\partial P_0}{\partial \gamma} \equiv h_0'' \end{aligned}\right\} \quad (78)$$

Adding in the terms multiplied by nt, these three equations may be written

$$D_{nt}l_0 = -h_0 - h_1 nt \qquad D_{nt}\pi_0 = -h_0' - h_1' nt \qquad D_{nt}\theta_0 = -h_0'' - h_1'' nt \quad (78')$$

The integration of (78) and (78′) will give

$$\begin{aligned} \delta\alpha &= \delta_0\alpha; & \delta l_0 &= \delta_0 l_0 - h_0 nt - \tfrac{1}{2}h_1 n^2 t^2 \\ \delta e &= \delta_0 e; & \delta\pi_0 &= \delta_0\pi_0 - h_0' nt - \tfrac{1}{2}h_1' n^2 t^2 \\ \delta\gamma &= \delta_0\gamma; & \delta\theta_0 &= \delta_0\theta_0 - h_0' nt - \tfrac{1}{2}h_1'' n^2 t^2 \end{aligned} \quad (78'')$$

δ_0 designating, in each case, the arbitrary constant of integration.

The completed expressions for l, π, and θ are to be found by the equations

$$\delta n = \frac{\partial n}{\partial \alpha}\delta\alpha \qquad \delta\pi_1 = \frac{\partial \pi_1}{\partial \alpha}\delta\alpha + \frac{\partial \pi_1}{\partial e}\delta e + \frac{\partial \pi_1}{\partial \gamma}\delta\gamma \qquad \delta\theta_1 = \frac{\partial \theta_1}{\partial \alpha}\delta\alpha + \frac{\partial \theta_1}{\partial e}\delta e + \frac{\partial \theta_1}{\partial \gamma}\delta\gamma \tag{79}$$

$$\delta l = \delta l_0 + \int \delta n dt \qquad \delta\pi = \delta\pi_0 + \int \delta\pi_1 dt \qquad \delta\theta = \delta\theta_0 + \int \delta\theta_1 dt \tag{80}$$

In these equations the perturbations (78″) are to be substituted. In doing this the arbitrary constants δl_0, $\delta_0\pi_0$, and $\delta_0\theta_0$, being merely constant corrections to l, π, and θ, may be dropped as unimportant to the theory. We shall then have from (78″) and (80)

$$\begin{aligned} \delta l &= \left(\frac{\partial n}{\partial \alpha}\delta_0\alpha - h_0 n\right)t - \tfrac{1}{2}h_1 n^2 t^2 \\ \delta\pi &= \left(\frac{\partial \pi_1}{\partial \alpha}\delta_0\alpha + \frac{\partial \pi_1}{\partial e}\delta_0 e + \frac{\partial \pi_1}{\partial \gamma}\delta_0\gamma - h_0' n\right)t - \tfrac{1}{2}h_1' n^2 t^2 \\ \delta\theta &= \left(\frac{\partial \theta_1}{\partial \alpha}\delta_0\alpha + \frac{\partial \theta_1}{\partial e}\delta_0 e + \frac{\partial \theta_1}{\partial \gamma}\delta_0\gamma - h_0 n\right)t - \tfrac{1}{2}h_1'' n^2 t^2 \end{aligned} \tag{81}$$

§ 29. *Adjustment of the arbitrary constants.* Values are next to be assigned to the arbitrary constants $\delta_0 a$, $\delta_0 e$, and $\delta_0\gamma$. We shall do this so as to satisfy the conditions that the coefficient of t in δl, of sin g in the mean longitude, and of sin $(l-\theta)$ in the latitude, shall all remain unchanged. The first of these conditions gives

$$-\tfrac{3}{2}n\delta_0\alpha = h_0 n \qquad \text{or} \qquad \delta_0\alpha = -\tfrac{2}{3}h_0 \tag{82}$$

We thus have

$$\delta l = -\tfrac{1}{2}h_1 n^2 t^2$$

The determination of $\delta_0 e$ and $\delta_0\gamma$ must await the computation of the periodic terms depending on the arguments g and $l-\theta$, which is found in Part IV. The increments in the motions of π and θ now become

$$\begin{aligned} \delta\pi_1 &= n\left(\frac{\partial \pi_1}{\partial n}h_0 - h_0'\right) + \frac{\partial \pi_1}{\partial e}\delta_0 e + \frac{\partial \pi_1}{\partial \gamma}\delta_0\gamma \\ \delta\theta_1 &= n\left(\frac{\partial \theta_1}{\partial n}h_0 - h_0''\right) + \frac{\partial \theta_1}{\partial e}\delta_0 e + \frac{\partial \theta_1}{\partial \gamma}\delta_0\gamma \end{aligned} \tag{83}$$

§ 30. *Opposite secular effects of the direct and indirect action of a planet near the Sun.*

An important theorem of the planetary action on the Moon is that as the planet is nearer the Sun, not only does each form of action become smaller, but the two forms tend to cancel each other, so that when the mass of the planet can be considered as simply added to that of the Sun, the non-parallactic perturbations vanish.

To find the effect of the direct action in this case, let the values of x_4, y_4, and z_4 in (14) be so small that they may be neglected in comparison with x', y', and z'. Then Δ will merge into r' and we shall have

$$R = \frac{m_4 r^2}{2r'^3}\left(3S^2 - 1\right)$$

For the indirect action we remark that the only effect of the action of the planet on the position of the Earth, after so adjusting the constants of integration that the mean motion shall remain unaltered, is to increase the mean distance, so that instead of

$$a'^3 n'^2 = m'$$

we shall have

$$a'^3 n'^2 = m' + m_4$$

This gives, for the perturbation of a'

$$3a'^2 n'^2 \delta a' = m_4$$

and the eccentricity e' being unaltered

$$\delta\rho' = \frac{m_4}{3m'}$$

The corresponding part of Ω_p is found from (7) by assigning the increment $r'\delta\rho'$ to r'. We thus have

$$\Omega_p = -\frac{3m' r^2 \delta\rho'}{2r'^3}(3S^2 - 1) = -\frac{m_4 r^2}{2r'^3}(3S^2 - 1)$$

This cancels the value of R found above.

PART II.

NUMERICAL DEVELOPMENT OF THE PLANETARY COEFFICIENTS.

CHAPTER IV.

COEFFICIENTS FOR THE DIRECT ACTION.

§ 31. In order to complete the expression for the disturbing forces, it is necessary to determine the numerical values of the coefficients A, B, etc., found in equation (14). I shall hereafter term these planetary coefficients. In all the investigations of the subject heretofore made the coördinates of the Sun and planets used in forming these coefficients have been taken as equal to their elliptic values, the effect of perturbations being neglected. In this case the methods of numerical development worked out by Hansen are in principle complete and satisfactory so far as purely elliptical values are concerned. They require the development of the inverse odd powers of the distance of the planet from the centre G, in a series proceeding according to the sines and cosines of multiples of the two mean longitudes. This being done, the formation of the powers and products of the planetary coördinates and their multiplication into the quantities Δ^{-3}, Δ^{-5}, etc., are mere matters of mechanical computation. It was in this way that the developments given in my former paper on the "Action of the Planets on the Moon" was effected.

A serious practical difficulty was, however, found in the fact that in the case of Venus and Mars the numerical products of the higher orders whose sums express A, B, etc., are far smaller than those which express Δ^{-3}, etc. Thus the final values of the quantities required come out as small differences between large numbers. An example is given on page 263 of the work referred to, where it is shown that the numerical sums of the positive and negative quantities, which form the coefficients for the planetary coefficients in the term of long period due to the action of Venus are more than 400 times as large as their difference, which is the coefficient finally to be used. The result is that long and intricate computations with large numbers have to be carried through in order to obtain a comparatively minute result.

The difficulty is lessened when, with Delaunay and Radau, the coefficients are found by algebraic formulæ through a development in powers of the eccentricities and inclination. But it is difficult to make sure that such a development is carried far enough, unless by the employment of very complex formulæ.

The difficulty is magnified many fold, in forming these perturbations, if account is taken of the mutual perturbations of the planets. While it might seem at first sight that these must be too small to affect the result, I showed in the former paper that we could not be sure of this point without actual computation. In the case of the Hansenian inequality of long period due to the action of Venus it was shown that the perturbations in question, considered individually, were nearly of the same order of magnitude as the coefficients to be determined. This proceeded from the fact that, even when we consider only the formulæ of the elliptic motion, the coefficients of the term in question are in the nature of minute residual differences of large quantities. In view of the undoubted fact of some apparent inequalities of long period in the motion of the Moon of which theory has yet given no explanation, it seems necessary to exhaustively discuss every possible mode of action which might affect the result.

The most effective and certain way which the author could devise to overcome this difficulty was to employ the purely numerical development sometimes called "mechanical quadratures," but, more exactly, that of induction of general formulæ from special values. It is true that the numerical computations required by this method would be very voluminous, possibly more so than those by other methods. But the use of the method has the great advantage that the computations are made on a simple and uniform plan, which can be executed by routine computers, and in which the complexity incident to the analytic treatment does not enter at all. Another important advantage of this purely numerical method is that the mutual periodic perturbations of Venus and the Earth can be taken account of from the beginning. This will readily be seen by a statement of the method.

The values of the planetary coefficients A, B, etc., being functions of the geocentric coördinates of Venus, can be computed for any assigned mean longitude of the Earth and Venus. They are therefore to be computed for a certain number of equidistant values of the mean longitude of each planet. For each of these values there will be a definite perturbation of the coördinates of each planet, which may be computed and applied in advance. Thus the first computation gives at once numerical values of the coefficients in which the effect of periodic perturbation is included. From these are developed by well-known formulæ the coefficients of the sines and cosines of the multiples of the mean longitudes.

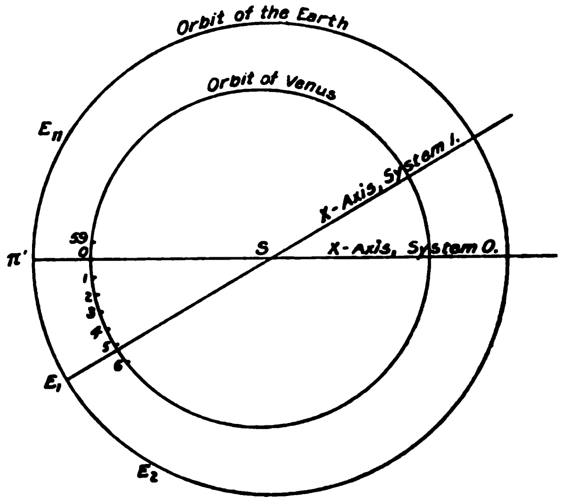
The perturbations of Mars are so small that it was assumed that undisturbed values of the coefficients would suffice. But the same method was used owing to its simplicity in theory.

In the case of Jupiter the analytic development would not have involved the difficulty which I have pointed out. But it was so convenient to apply the numerical method that it was adopted for this planet also.

The action of Saturn is so minute that a very simple development suffices. It was therefore unnecessary to employ the numerical method in this case.

A. Action of Venus.

§ 32. We shall now show how the computations were arranged in the case of Venus. Let us first suppose that the orbits of both planets are circular. Then assume the Earth to be in zero of longitude. We assign in succession 60 equidistant longitudes to Venus, 6° apart. For each of these positions we compute the values of the four principal coefficients. Numerical induction from these special values will then give the values of A, B, etc., in a series proceeding according to the cosines of the multiples of the differences of the mean longitudes.



Arrangement of Coördinate Axes, in Systems 0, 1, etc., for Venus.

Now assign to the Earth a mean longitude equal to any multiple of 6°. If we start with Venus at inferior conjunction we shall have the same series of values of the coefficients as before, provided that we now take the line joining the Sun and Earth as the axis of X. Supposing all our coördinates referred to this axis we should then have A, B, etc., developed according to cosines of multiples of the difference of the mean longitudes.

It follows that in the actual case of the two orbits having a small eccentricity and inclination the other terms which we require will be of the order of magnitude of these quantities and will therefore be smaller than these principal terms. It is therefore not necessary to divide the circle into so many parts in order to obtain them.

The actual process was to take the direction of the solar perigee for 1800 as the initial line, or axis of X. The way in which the coördinates were defined will then be seen by the diagram. Here on the left, π' marks the position of the Earth's perihelion. The positive direction of X passes through the Sun and is therefore directed toward the solar perigee. The Earth being in this fixed position, the coördinates of Venus are computed for 60 equidistant values of the mean longitude of Venus differing by increments of 6°. The initial or zero value corresponds to the mean inferior conjunction of Venus, marked 0 in the figure, which determines all the other values; a few of the others are numbered in order.

For each of these mean longitudes, the actual coördinates of Venus, including the effect of perturbations by the Earth, were computed. The position of the Earth at π', corresponding to the 6° positions of Venus, was then corrected in each of the 60 cases by the periodic perturbations due to each position of Venus. With these coördinates 60 numerical values of the A-coefficients are computed. I designate this system of 60 values, corresponding, perturbations aside, to one position of the Earth, by the number 0; and I distinguish the values by 60 indices 0, 1, 2, . . . 59.

In the next system, called system 1, the Earth has moved through 30° of mean longitude, or mean anomaly, to the position E_1. The set of 60 heliocentric coördinates of Venus to be used will be the same as before, except for the perturbations, which will now be those for E_1, or for $g=30°$. But the position corresponding to the inferior conjunction in this system will be that corresponding to the index 5 in system 0. A new axis of X is now adopted, again passing through the mean Sun, and therefore making an angle of 30° with the initial axis. The coördinates of Venus are all transformed to this axis, and another set of 60 values of the A-coefficients are computed.

The remainder of the process consists in assigning to the mean longitude of the Earth successive increments of 30° until it is brought around to the position E_{11}, in mean anomaly 330°. In each case the axis of X is taken to pass through the mean Sun.

From these 720 special values of the A-coefficients the general values are separately developed for each of the 12 systems. Then the general development for any system is effected by a second quadrature. The final result will be the values of A, B, etc., referred to an axis always passing through the mean Sun.

Were we to adopt a fixed system of coördinate axes, it would now be necessary to transform these values referred to the moving axis, to the adopted fixed system. But the necessity of this transformation is avoided by referring all the coördinates, those of the Moon as well as of the planet, to the mean Sun from the beginning. This is fully as simple as, perhaps even simpler than, referring them to a fixed axis. The ease of doing it is all the greater from the fact that, in the actual computation of the lunar coördinates, they are first referred to the mean Moon. The transformation from the mean Moon to the mean Sun is probably simpler than the transformation to a fixed axis.

§ 33. *Development of the A-coefficients for Venus.*

The computations relating to Venus are shown in tabular form in Tables I–VIII, and will now be explained. To obtain the 12 undisturbed values of the Sun's coördinates, we derive the equation of the centre and the logarithm of the radius vector from the tables of the Sun found in *Astronomical Papers*, Vol. VI.

For the argument of mean anomaly of the Sun the initial value is

$$M_0 = 5.37018$$

corresponding to $g' = 0°$. The increment for each 30° is

$$\Delta M = 30.43830$$

resulting in the value 188.0000 for $g' = 180°$.

With the 12 values of M thus found are taken the equation of the centre, E, for 1800, and log r'. Then

$$x' = r' \cos E \qquad y' = r' \sin E$$

The resulting values of x' and y' are shown in Table I.

For Venus, we have

$$\text{Initial mean longitude} = 99^\circ\, 30'\, 7''$$

this being the longitude of the Earth's perihelion for 1800. For the same epoch we have

$$\begin{array}{ll} \text{Longitude of perihelion of Venus} & = 128^\circ\, 45'\, 17''.4 \\ \text{Initial mean anomaly of Venus} & = 330^\circ\, 44'\, 50''.2 \end{array}$$

To find the tabular argument corresponding to this mean anomaly we proceed thus: Adding 5 increments of 6°, we have

$$\text{Mean anomaly of Venus for index } 5 = 0^\circ\, 44'\, 50''.25$$

For this mean anomaly the precepts of *Tables of Venus*, pp. 278–279, give

$$\begin{array}{l} \text{Tabular Arg. } K\,; \quad K_0 = 1.11601 \\ \text{Increment of } K \text{ for } 6^\circ = 3.745014 \end{array}$$

We now add one period to K and subtract 5 increments

$$\begin{array}{rr} K_5 = & 1.11601 \\ P = & \underline{224.70084} \\ & 225.81685 \\ \text{5 increments} & \underline{18.72507} \\ \text{Initial } K, & 207.09178 \end{array}$$

which corresponds to the inferior conjunction of Venus in system 0.

The resulting values of K are found in Table II. With the values of K thus formed the equation of the centre and log r in the elliptic orbit of 1800 are taken from the tables.

The data for the rectangular coördinates are:

Node of Venus, 1800	$\Omega =$	74° 52′ 48″.75
Perihelion of Earth	$\pi =$	99 30 7 .6
Node referred to Perihelion	$\theta =$	335 22 41 .2
Inclination for 1800	$\gamma =$	3 23 33 .45

The values of the coördinates x, y, and z in the initial system are now computed by the formulæ

$$u = \text{Eq. Cent.} + 24^\circ\, 37'\, 18''.80 \qquad m \sin M = \cos\gamma \sin\theta \qquad m \cos M = \cos\theta$$

$$m' \sin M' = \sin\theta \qquad m' \cos M' = \cos\gamma \cos\theta$$

leading to

$$\begin{array}{ll} M = 24^\circ\, 35'\, 1''.84 & M' = 24^\circ\, 39'\, 35''.91 \\ \log m = 9.9998680 & \log m' = 9.9993706 \end{array}$$

$$x = mr \cos (M + u) \qquad y = m'r \sin (M' + u) \qquad z = r \sin\gamma \sin u$$

Designating the systems by suffixes, and putting $c = \cos 30°$, these coördinates were transformed to the axes of the other 11 systems by the formulæ

$$\begin{array}{ll} x_1 = cx_0 + \tfrac{1}{2}y_0 & y_1 = cy_0 - \tfrac{1}{2}x_0 \\ x_2 = \tfrac{1}{2}x_0 + cy_0 & y_2 = \tfrac{1}{2}y_0 - cx_0 \\ x_3 = y_0 & y_3 = -x_0 \\ x_4 = y_1 & y_4 = -x_1 \\ x_5 = y_2 & y_5 = -x_2 \end{array}$$

and then, in general,

$$x_n = -x_{n-6} \qquad y_n = -y_{n-6}$$

§ 34. *Explanation of the tables.* The periodic perturbations of the longitudes of the Earth and Venus, and of the logarithms of their radii vectores, omitting terms of long period, are now to be found.

TABLE III: *Mutual periodic perturbations.*

For the perturbations of Venus by the Earth, δu and $\delta \rho$, the arguments of the double entry Tables VIII and XVII are:

Hor. Arg. $g = K - 0^d.650 = 206^d.44 + 3.745i$
Vert. Arg. II for System 0 and $i = 0$, 104.35
Increment of II for each system $\Delta\text{II} = 20$
" " " " " index $\Delta_2\text{II} = -2.461$

For the single entry Tables XI and XX we have

$$\text{Arg. } A = 1.62203(g - g')$$

For the index i $g = 330°.75 + 6°i$
For the jth system $g' = 30°j$

Hence, for $i = 0$, $j = 0$, Arg. $A = 536.49$
Increment for each unit of i, $\Delta A = +9.732$
" " " " " j, $\Delta' A = -48.661$

With the values of the arguments thus formed the periodic perturbations of Venus by the Earth are taken from the Tables VIII, XVII, XI, and XX.

For the corresponding perturbations of the Earth by Venus, we have

Hor. Arg. $g = 30.4383j$
Vert. Arg. II for $j = 0$; $i = 0 \cdots 165.375$
Increment for each unit of i; $\tfrac{1}{2}\Delta g = 3$
" " " " " j; -24.383

Argument A has the same value as in the *Tables of Venus.*

The perturbations of the longitude and log. radius vector of the Earth found with these values of the arguments are given in the columns $\delta v'$ and $\delta \rho'$.

TABLE IV*a* AND IV*b*. The perturbations in Table III are transformed into increments of the rectangular coördinates of Venus and the Earth.

Neglecting the cosine of the inclination we have for Venus when referred to the initial system of axes

$$\Delta x_0 = -y \sin 1'' \delta u + x \delta \rho \qquad \Delta y_0 = x \sin 1'' \delta u + y \delta \rho$$

the tabular $\delta \rho$ being multiplied by the modulus of logarithms. For the other systems the transformation is made by the formulæ for the transformation of the coördinates themselves. The results are given in full, in units of the 8th place of decimals, in Table IV. Applying them to the undisturbed coördinates, we have the coördinates of Venus for each position of the two bodies.

TABLE V. The values of the solar coördinates in Table I, of the Venus coördinates in Table II, after being transformed to the axis of the system, and of the increments in Table IV, are added so as to form the disturbed geocentric coördinates of Venus in each system for each position of Venus.

TABLE VI. With the perturbations of latitude in the different systems the disturbed geocentric coördinate Z was computed and tabulated.

With these geocentric coördinates are computed the 720 values of the four coefficients A, B, C, and D defined in § 7. Since

$$A + B + C = 0$$

the computation of C might have been dispensed with. It was, however, carried through as an additional check on the accuracy of the work. The latter was, however, done in duplicate, the check being incomplete.

TABLE VII gives the values of the coefficients thus computed.

The coefficients E and F lead to appreciable inequalities only in the case of the argument θ, and have been treated separately. Their special values were computed for six systems and thirty indices only, and are found in Table VIII.

§ 35. The process of developing the general value of each coefficient in a periodic series is given by Brünnow in his *Sphärischen Astronomie.* Taking A as an example we first develop the value for each system in the form

$$A_k = \Sigma (a_k \cos iL + b_k \sin iL)$$

where k is the number of the system and L the difference of the mean longitudes of Venus and the Earth,

$$L = \text{v} - g'$$

We thus have 12 values of each of the coefficients a_k and b_k, one corresponding to each value of g'. These values are then again developed in the form

$$a_k = \Sigma (a_{k,j} \cos jg' + b_{k,j} \sin jg') \qquad b_k = \Sigma (a_{k,j}' \cos ig' + b_{k,j}' \sin jg')$$

These being substituted in the general expression given above for A_k gives the value of A itself in the form

$$A = \Sigma\Sigma\,[a \cos(iL + jg') + b \sin(iL + jg')]$$

The development was effected in this way up to $i = 8$ only, this being the limit for possible sensible terms other than the Hansenian term of long period depending on the argument

$$18L + 2g' - g$$

§ 36. *The Hansenian Venus-term of long period.* The computation of this inequality requires the determination of the coefficients for $i = 18$, which we obtain from the general formulæ thus. Putting, in any one system,

$$A_0, A_1, A_2, \cdots A_{59}$$

for the 60 values of A, and

$$A_c \cos 18L + A_s \sin 18L$$

for the pair of terms depending on the argument $18L$, the general formulæ give

$$30A_c = A_0 + A_1 \cos 108^\circ + A_2 \cos 216^\circ + \cdots$$
$$30A_s = A_1 \sin 108^\circ + A_2 \sin 216^\circ + \cdots$$

the angles increasing by 108° in each term. The fifth angle will be $180^\circ + 2\pi$, so that the only numerically different values of the coefficients which enter into the series besides 1 and 0 are

$$\sin 18^\circ,\ \cos 18^\circ,\ \sin 36^\circ, \text{ and } \cos 36^\circ$$

For example, we have

$$30A_c = A_0 - A_1 \sin 18^\circ - A_2 \cos 36^\circ + A_3 \cos 36^\circ + A_4 \sin 18^\circ + \cdots$$
$$30A_s = A_1 \cos 18^\circ - A_2 \sin 36^\circ - A_3 \sin 36^\circ + A_4 \cos 18^\circ + \cdots$$

From the cyclic order of the coefficients the method of computing A_c and A_s is as follows:

With the 60 values of any one coefficient, say A, in any one system,

$$A_0, A_1, A_2, \cdots, A_{59}$$

compute

$$A_0' = A_0 + A_{10} + A_{20} + \cdots + A_{50}$$
$$A_1' = A_1 + A_{11} + A_{21} + \cdots + A_{51}$$
$$A_2' = A_2 + A_{12} + A_{22} + \cdots + A_{52}$$
$$\vdots$$
$$A_9' = A_9 + A_{19} + A_{29} + \cdots + A_{59}$$

Next:

$$A_0''=A_0'-A_5' \qquad A_1''=A_1'-A_6' \qquad A_2''=A_2'-A_7' \qquad A_3''=A_3'-A_8' \qquad A_4''=A_4'-A_9'$$

Next:

$$A_{s,1}=A_4''+A_1'' \qquad A_{c,1}=A_4''-A_1'' \qquad A_{s,1}=A''+A_2'' \qquad A_{c,2}=A_3''-A_2''$$

We then have, in each system

$$30A_c=A_0''+A_{c,1}\sin 18^\circ+A_{c,2}\cos 36^\circ \qquad 30A_s=A_{s,1}\cos 18^\circ-A_{s,2}\sin 36^\circ$$

with similar values for B, C, and D.

The numerical results of these processes for each system are shown in Table IX.

The next step is to develop each set of numerical values of any one pair of coefficients, say A_c, and A_s in the form

$$\begin{aligned}30A_c&=\alpha_0+\alpha_1\cos g'+\alpha_2\cos 2g'+\beta_1\sin g'+\beta_2\sin 2g'\\ 30A_s&=\alpha_0'+\alpha_1'\cos g'+\alpha_2'\cos 2g'+\beta_1'\sin g'+\beta_2'\sin 2g'\end{aligned} \qquad (a)$$

These are to be substituted in the general form

$$A=A_s\cos 18L+A_s\sin 18L$$

Retaining only terms which may be wanted for our purpose, we shall have

$$\begin{aligned}30A&=\alpha_0\cos 18L+\alpha_0'\sin 18L\\ &+\tfrac{1}{2}(\alpha_1-\beta_1')\cos(18L+g')+\tfrac{1}{2}(\alpha_1'+\beta_1)\sin(18L+g')\\ &+\tfrac{1}{2}(\alpha_2-\beta_2')\cos(18L+2g')+\tfrac{1}{2}(\alpha_2'+\beta_2)\sin(18L+2g')\\ &+\tfrac{1}{2}(\alpha_3-\beta_3')\cos(18L+3g')+\tfrac{1}{2}(\alpha_3'+\beta_3)\sin(18L+3g')\end{aligned} \qquad (b)$$

TABLE I.

SUN'S GEOCENTRIC COÖRDINATES IN THE MEAN ORBIT OF 1800, REFERRED TO MEAN SUN AS DIRECTION OF AXIS OF X.

System.	g'	x'	y'
	°		
0	0	+0.983 2075	0.000 0000
1	30	+0.985 3853	+0.016 8542
2	60	+0.991 3897	+0.029 1452
3	90	+0.999 7183	+0.033 5823
4	120	+1.008 1877	+0.029 0233
5	150	+1.014 4741	+0.016 7321
6	180	+1.016 7929	0.000 0000
7	210	+1.014 4741	−0.016 7321
8	240	+1.008 1877	−0.029 0233
9	270	+0.999 7183	−0.033 5823
10	300	+0.991 3897	−0.029 1452
11	330	+0.985 3853	−0.016 8542

TABLE II.

COMPUTATION OF RECTANGULAR COÖRDINATES OF VENUS IN THE ELLIPTIC ORBIT OF 1800, REFERRED TO SOLAR PERIGEE AS AXIS OF X.

i	Arg. K.	Eq. Cent.		log. r	log. x	log. y	log. z
		′	″				
0	207.0918	−23	15.02	9.856 7321	−0.856 5920	+7.726 9130	+8.242 1742
1	210.8368	−18	47.49	9.856 5914	−9.854 2879	−8.849 0901	+8.331 7492
2	214.5818	−14	7.30	9.856 4812	−9.847 0688	−9.163 8652	+8.401 8640
3	218.3268	− 9	17.60	9.856 4028	−9.834 7638	−9.341 2292	+8.457 9729
4	222.0718	− 4	21.63	9.856 3570	−9.817 0717	−9.463 1513	+8.503 3027
5	1.1160	+ 0	37.28	9.856 3445	−9.793 5309	−9.554 3215	+8.539 9034
6	4.8610	+ 5	35.76	9.856 3653	−9.703 4657	−9.625 5304	+8.569 1404
7	8.6060	+10	30.48	9.856 4193	−9.725 9000	−9.682 4522	+8.591 9486
8	12.3510	+15	18.10	9.856 5057	−9.679 4042	−9.728 4252	+8.608 9752
9	16.0961	+19	55.43	9.856 6236	−9.621 8221	−9.765 5624	+8.620 6617
10	19.8411	+24	19.34	9.856 7717	−9.549 7350	−9.795 2686	+8.627 2934
11	23.5861	+28	26.93	9.856 9481	−9.457 3022	−9.818 5057	+8.629 0280
12	27.3311	+32	15.39	9.857 1509	−9.333 3365	−9.835 9401	+8.625 9108
13	31.0761	+35	42.29	9.857 3778	−9.151 9096	−9.848 0289	+8.617 8781
14	34.8211	+38	45.29	9.857 6261	−8.824 3320	−9.855 0707	+8.604 7511
15	38.5661	+41	22.44	9.857 8931	+7.961 5592	−9.857 2356	+8.586 2194
16	42.3112	+43	32.07	9.858 1756	+8.929 0946	−9.854 5813	+8.561 8070
17	46.0562	+45	12.78	9.858 4706	+9.203 5288	−9.847 0574	+8.530 8192
18	49.8012	+46	23.54	9.858 7747	+9.367 1085	−9.834 5015	+8.492 2492
19	53.5462	+47	3.66	9.859 0846	+9.482 0030	−9.816 6224	+8.444 6187
20	57.2912	+47	12.78	9.859 3968	+9.568 8886	−9.792 9712	+8.385 6881
21	61.0362	+46	50.89	9.859 7079	+9.637 2113	−9.762 8896	+8.311 8851
22	64.7812	+45	58.29	9.860 0146	+9.692 0696	−9.725 4250	+8.217 0551
23	68.5263	+44	35.65	9.860 3134	+9.736 5149	−9.679 1841	+8.089 2497
24	72.2713	+42	43.92	9.860 6011	+9.772 5022	−9.622 0691	+7.900 1224
25	76.0163	+40	24.44	9.860 8748	+9.801 3428	−9.550 7649	+7.546 9461
26	79.7613	+37	38.73	9.861 1314	+9.823 9406	−9.459 6400	−6.971 8246
27	83.5063	+34	28.65	9.861 3684	+9.840 9243	−9.338 0001	−7.731 3881
28	87.2513	+30	56.31	9.861 5830	+9.852 7264	−9.161 4423	−7.990 3231
29	90.9963	+27	4.01	9.861 7733	+9.859 6294	−8.849 5051	−8.148 1795
30	94.7414	+22	54.28	9.861 9370	+9.861 7933	+7.639 3795	−8.260 1046
31	98.4864	+18	29.84	9.862 0726	+9.859 2710	+8.899 7388	−8.345 1900
32	102.2314	+13	53.52	9.862 1786	+9.852 0116	+9.186 2719	−8.412 3233
33	105.9764	+ 9	8.29	9.862 2540	+9.839 8567	+9.354 2638	−8.466 3544
34	109.7214	+ 4	17.19	9.862 2779	+9.822 5254	+9.471 5502	−8.510 2060
35	113.4664	− 0	36.64	9.862 3099	+9.799 5863	+9.560 0170	−8.545 7583
36	117.2114	− 5	30.07	9.862 2809	+9.770 4104	+9.629 5228	−8.574 2757
37	120.9564	−10	20.01	9.862 2381	+9.734 0928	+9.685 3431	−8.596 6294
38	124.7015	−15	3.29	9.862 1550	+9.689 3188	+9.730 6171	−8.613 4244
39	128.4465	−19	36.93	9.862 0415	+9.634 1256	+9.767 3464	−8.625 0747
40	132.1915	−23	57.98	9.861 8988	+9.565 4464	+9.796 8687	−8.631 8465
41	135.9365	−28	3.60	9.861 7283	+9.478 1499	+9.820 1022	−8.633 8851
42	139.6815	−31	51.16	9.861 5318	+9.362 7126	+9.837 6839	−8.631 2325
43	143.4265	−35	18.17	9.861 3113	+9.198 2317	+9.850 0500	−8.623 8210
44	147.1715	−38	22.35	9.861 0693	+8.921 4921	+9.857 4837	−8.611 4784
45	150.9166	−41	1.70	9.860 8081	+7.912 5788	+9.860 1443	−8.593 9051
46	154.6616	−43	14.42	9.860 5307	−8.827 3540	+9.858 0817	−8.570 6520
47	158.4066	−44	58.99	9.860 2398	−9.151 8035	+9.851 2413	−8.541 0589
48	162.1516	−46	14.23	9.859 9388	−9.332 3188	+9.839 4597	−8.504 1837
49	165.8966	−46	59.22	9.859 6307	−9.455 9020	+9.822 4495	−8.458 6539
50	169.6416	−47	13.42	9.859 3190	−9.548 1734	+9.799 7714	−8.402 4145
51	173.3866	−46	56.56	9.859 0071	−9.620 2216	+9.770 7859	−8.332 2360
52	177.1317	−46	8.76	9.858 6983	−9.677 8438	+9.734 5737	−8.242 6677
53	180.8767	−44	50.45	9.858 3962	−9.724 4363	+9.689 7990	−8.123 4525
54	184.6217	−43	2.43	9.858 1040	−9.762 1421	+9.634 4660	−7.951 5304
55	188.3667	−40	45.82	9.857 8250	−9.792 3808	+9.565 4503	−7.653 3423
56	192.1117	−38	2.06	9.857 5624	−9.816 1217	+9.477 5085	−4.949 3707
57	195.8567	−34	52.90	9.857 3192	−9.834 0339	+9.360 8616	+7.651 6300
58	199.6017	−31	20.43	9.857 0981	−9.846 5740	+9.193 8645	+7.950 6732
59	203.3468	−27	26.94	9.856 9016	−9.854 0381	+8.909 8324	+8.122 8613

TABLE III.

Mutual periodic Perturbations of Venus and the Earth.

The term of long period is omitted. The tabular unit is 0''.01 in δu and $\delta v'$, and 10^{-8} in $\delta\rho$ and $\delta\rho'$.

i	System 0.				System 1.				System 2.			
	δu	$\delta v'$	$\delta\rho$	$\delta\rho'$	δu	$\delta v'$	$\delta\rho$	$\delta\rho'$	δu	$\delta v'$	$\delta\rho$	$\delta\rho'$
0	+ 239	−169	+ 347	+464	+ 534	−390	+ 321	+472	+ 543	−402	+ 229	+537
1	+ 65	−151	+ 394	+440	+ 432	−423	+ 398	+424	+ 545	−507	+ 331	+467
2	− 146	−116	+ 515	+365	+ 267	−423	+ 540	+334	+ 479	−573	+ 514	+349
3	− 406	− 65	+ 675	+254	+ 23	−379	+ 716	+214	+ 315	−590	+ 724	+203
4	− 705	+ 4	+ 814	+126	− 295	−303	+ 880	+ 78	+ 58	−558	+ 910	+ 42
5	−1018	+ 82	+ 897	+ 0	− 662	−194	+ 988	− 62	− 276	−477	+1045	−117
6	−1304	+158	+ 898	−111	−1052	− 62	+1023	−194	− 659	−357	+1093	−261
7	−1541	+219	+ 823	−206	−1420	+ 80	+ 941	−308	−1058	−204	+1044	−388
8	−1702	+264	+ 687	−284	−1728	+221	+ 779	−397	−1435	− 27	+ 895	−489
9	−1796	+289	+ 501	−352	−1941	+348	+ 551	−459	−1760	+161	+ 658	−562
10	−1820	+301	+ 281	−414	−2049	+449	+ 273	−495	−1991	+348	+ 349	−602
11	−1777	+307	+ 26	−468	−2047	+522	− 21	−514	−2103	+517	+ 5	−612
12	−1674	+313	− 234	−510	−1950	+566	− 316	−518	−2089	+657	− 343	−594
13	−1508	+325	− 506	−538	−1767	+589	− 596	−515	−1956	+757	− 661	−557
14	−1286	+343	− 767	−549	−1517	+599	− 851	−504	−1721	+818	− 935	−506
15	−1013	+370	−1002	−544	−1211	+605	−1066	−485	−1417	+847	−1146	−447
16	− 695	+403	−1205	−524	− 857	+607	−1236	−454	−1057	+850	−1298	−387
17	− 350	+446	−1361	−486	− 475	+610	−1348	−406	− 666	+836	−1386	−322
18	+ 19	+498	−1462	−433	− 77	+611	−1399	−346	− 260	+812	−1407	−255
19	+ 407	+559	−1486	−353	+ 320	+613	−1382	−269	+ 149	+781	−1357	−181
20	+ 787	+622	−1417	−247	+ 701	+614	−1302	−178	+ 535	+741	−1236	− 99
21	+1139	+681	−1244	−113	+1047	+616	−1148	− 72	+ 883	+695	−1053	− 6
22	+1431	+723	− 966	+ 40	+1351	+617	− 921	+ 50	+1182	+643	− 825	+ 96
23	+1629	+740	− 597	+205	+1586	+611	− 619	+189	+1417	+586	− 541	+209
24	+1709	+726	− 180	+367	+1725	+590	− 248	+338	+1578	+524	− 226	+328
25	+1660	+682	+ 263	+516	+1748	+550	+ 164	+489	+1652	+458	+ 121	+453
26	+1489	+606	+ 694	+646	+1636	+484	+ 586	+630	+1623	+380	+ 488	+578
27	+1215	+504	+1074	+752	+1394	+395	+ 973	+750	+1478	+288	+ 846	+695
28	+ 854	+376	+1381	+836	+1045	+285	+1287	+844	+1212	+179	+1162	+799
29	+ 438	+226	+1583	+890	+ 616	+160	+1503	+900	+ 842	+ 56	+1399	+875
30	− 9	+ 58	+1667	+912	+ 148	+ 24	+1608	+922	+ 397	− 77	+1532	+918
31	− 451	−122	+1616	+898	− 321	−122	+1596	+909	− 79	−214	+1544	+923
32	− 859	−301	+1439	+843	− 764	−272	+1463	+863	− 549	−342	+1437	+887
33	−1197	−468	+1142	+755	−1145	−426	+1210	+784	− 971	−467	+1213	+815
34	−1440	−610	+ 765	+637	−1429	−571	+ 860	+673	−1309	−581	+ 896	+709
35	−1577	−718	+ 336	+502	−1598	−700	+ 436	+536	−1540	−684	+ 501	+576
36	−1605	−793	− 103	+355	−1636	−803	− 14	+377	−1643	−777	+ 66	+420
37	−1529	−833	− 524	+204	−1557	−870	− 453	+206	−1610	−849	− 378	+247
38	−1357	−837	− 911	+ 51	−1360	−898	− 856	+ 33	−1443	−893	− 797	+ 67
39	−1058	−811	−1221	−102	−1085	−889	−1183	−132	−1163	−907	−1148	−115
40	− 770	−755	−1440	−248	− 747	−844	−1422	−284	− 800	−883	−1404	−288
41	− 393	−675	−1548	−380	− 370	−770	−1557	−418	− 389	−824	−1551	−443
42	− 8	−575	−1540	−485	+ 19	−669	−1579	−532	+ 31	−733	−1578	−573
43	+ 360	−467	−1430	−559	+ 395	−547	−1491	−620	+ 426	−620	−1503	−673
44	+ 681	−359	−1243	−599	+ 730	−410	−1303	−678	+ 780	−487	−1328	−741
45	+ 950	−264	−1003	−608	+1002	−269	−1035	−702	+1068	−340	−1067	−775
46	+1160	−187	− 739	−593	+1192	−135	− 731	−688	+1273	−188	− 750	−772
47	+1318	−130	− 463	−561	+1302	− 20	− 415	−641	+1383	− 38	− 403	−732
48	+1424	− 92	− 186	−518	+1347	+ 68	− 106	−566	+1399	+ 99	− 59	−656
49	+1485	− 69	+ 72	−465	+1334	+127	+ 148	−476	+1335	+212	+ 240	−552
50	+1497	− 59	+ 304	−398	+1284	+155	+ 367	−374	+1212	+291	+ 472	−422
51	+1467	− 61	+ 490	−319	+1206	+160	+ 537	−268	+1066	+331	+ 627	−277
52	+1401	− 74	+ 641	−232	+1111	+148	+ 655	−163	+ 904	+336	+ 709	−129
53	+1306	− 95	+ 739	−136	+1007	+116	+ 719	− 55	+ 757	+308	+ 739	+ 14
54	+1191	−122	+ 783	− 38	+ 903	+ 65	+ 731	+ 52	+ 634	+254	+ 701	+144
55	+1050	−147	+ 777	+ 63	+ 799	+ 1	+ 688	+161	+ 535	+182	+ 627	+264
56	+ 889	−165	+ 717	+168	+ 728	− 76	+ 603	+267	+ 467	+ 92	+ 504	+371
57	+ 723	−175	+ 611	+272	+ 670	−162	+ 494	+362	+ 443	− 17	+ 390	+461
58	+ 556	−179	+ 488	+367	+ 630	−251	+ 388	+435	+ 462	−142	+ 276	+527
59	+ 398	−177	+ 387	+436	+ 591	−329	+ 320	+474	+ 501	−274	+ 212	+556

TABLE III.—*Continued.*

MUTUAL PERIODIC PERTURBATIONS OF VENUS AND THE EARTH.

The term of long period is omitted. The tabular unit is $0''.01$ in δu and $\delta v'$, and 10^{-8} in $\delta\rho$ and $\delta\rho'$.

i	System 3.				System 4.				System 5.			
	δu	$\delta v'$	$\delta\rho$	$\delta\rho'$	δu	$\delta v'$	$\delta\rho$	$\delta\rho'$	δu	$\delta v'$	$\delta\rho$	$\delta\rho'$
0	+ 391	− 295	+ 106	+639	+ 249	− 194	− 20	+749	+ 110	− 87	− 121	+839
1	+ 452	− 431	+ 220	+555	+ 326	− 337	+ 89	+669	+ 223	− 253	− 31	+777
2	+ 456	− 543	+ 424	+419	+ 349	− 457	+ 293	+533	+ 273	− 391	+ 170	+649
3	+ 375	− 619	+ 667	+251	+ 299	− 548	+ 547	+357	+ 245	− 493	+ 424	+477
4	+ 197	− 647	+ 893	+ 66	+ 170	− 604	+ 802	+156	+ 128	− 556	+ 665	+277
5	− 68	− 623	+1059	−118	− 38	− 621	+1008	− 53	− 65	− 579	+ 903	+ 66
6	− 393	− 551	+1132	−287	− 310	− 596	+1132	−251	− 314	− 568	+1053	−144
7	− 744	− 435	+1106	−436	− 614	− 529	+1145	−430	− 595	− 524	+1111	−344
8	−1097	− 283	+ 977	−557	− 924	− 421	+1042	−576	− 882	− 447	+1062	−523
9	−1424	− 103	+ 760	−646	−1208	− 278	+ 838	−688	−1148	− 339	+ 902	−672
10	−1690	+ 97	+ 463	−701	−1441	− 108	+ 547	−761	−1362	− 202	+ 637	−781
11	−1876	+ 305	+ 109	−720	−1604	+ 79	+ 200	−799	−1499	− 44	+ 293	−845
12	−1958	+ 507	− 271	−702	−1688	+ 276	− 180	−799	−1545	+ 126	− 95	−863
13	−1919	+ 692	− 647	−654	−1681	+ 476	− 563	−765	−1501	+ 303	− 489	−844
14	−1761	+ 843	− 977	−578	−1582	+ 667	− 921	−695	−1372	+ 478	− 859	−788
15	−1505	+ 952	−1226	−487	−1393	+ 839	−1219	−594	−1174	+ 648	−1171	−701
16	−1173	+1014	−1378	−388	−1120	+ 979	−1433	−471	− 914	+ 805	−1412	−588
17	− 800	+1035	−1458	−287	− 787	+1078	−1541	−332	− 610	+ 944	−1554	−449
18	− 412	+1019	−1448	−188	− 429	+1127	−1539	−194	− 284	+1051	−1588	−291
19	− 23	+ 980	−1366	− 95	− 69	+1129	−1431	− 62	+ 45	+1121	−1503	−125
20	+ 344	+ 922	−1219	− 5	+ 260	+1089	−1245	+ 56	+ 347	+1143	−1313	+ 37
21	+ 679	+ 853	−1014	+ 84	+ 547	+1021	− 999	+162	+ 595	+1120	−1036	+188
22	+ 967	+ 774	− 758	+172	+ 788	+ 931	− 713	+254	+ 778	+1056	− 706	+313
23	+1195	+ 682	− 466	+262	+ 975	+ 827	− 403	+340	+ 893	+ 959	− 356	+416
24	+1349	+ 582	− 151	+353	+1102	+ 712	− 81	+419	+ 942	+ 841	− 7	+495
25	+1428	+ 477	+ 171	+446	+1161	+ 587	+ 241	+492	+ 936	+ 710	+ 322	+557
26	+1427	+ 364	+ 489	+542	+1148	+ 449	+ 546	+558	+ 875	+ 569	+ 623	+609
27	+1339	+ 248	+ 786	+635	+1067	+ 307	+ 817	+617	+ 765	+ 422	+ 879	+650
28	+1164	+ 126	+1059	+726	+ 920	+ 163	+1043	+674	+ 608	+ 258	+1086	+684
29	+ 903	− 4	+1273	+804	+ 719	+ 13	+1208	+726	+ 412	+ 107	+1216	+707
30	+ 562	− 143	+1416	+860	+ 473	− 138	+1313	+770	+ 194	− 56	+1278	+720
31	+ 160	− 288	+1454	+886	+ 186	− 290	+1341	+803	− 36	− 219	+1263	+726
32	− 273	− 429	+1379	+876	− 137	− 442	+1282	+814	− 269	− 379	+1177	+720
33	− 692	− 562	+1183	+829	− 472	− 591	+1117	+796	− 491	− 536	+1003	+707
34	−1052	− 675	+ 887	+743	− 781	− 730	+ 854	+742	− 692	− 685	+ 773	+676
35	−1325	− 764	+ 514	+625	−1038	− 848	+ 500	+652	− 857	− 824	+ 461	+624
36	−1484	− 832	+ 96	+479	−1206	− 937	+ 94	+526	− 966	− 948	+ 93	+540
37	−1523	− 878	− 334	+311	−1269	− 991	− 329	+376	−1002	−1047	− 309	+425
38	−1427	− 902	− 751	+129	−1221	−1007	− 740	+207	− 945	−1106	− 718	+281
39	−1212	− 906	−1112	− 59	−1065	− 991	−1095	+ 26	− 797	−1124	−1082	+114
40	− 888	− 884	−1392	−244	− 809	− 946	−1376	−158	− 564	−1095	−1373	− 65
41	− 482	− 837	−1560	−417	− 477	− 880	−1558	−339	− 267	−1026	−1562	−242
42	− 44	− 759	−1603	−567	− 89	− 793	−1622	−507	+ 61	− 920	−1635	−411
43	+ 392	− 655	−1524	−689	+ 322	− 685	−1563	−654	+ 408	− 791	−1586	−564
44	+ 778	− 530	−1338	−777	+ 715	− 558	−1385	−769	+ 743	− 643	−1428	−696
45	+1095	− 388	−1067	−828	+1054	− 418	−1106	−846	+1043	− 484	−1159	−800
46	+1319	− 240	− 749	−841	+1304	− 268	− 762	−880	+1279	− 319	− 819	−869
47	+1445	− 89	− 392	−815	+1453	− 114	− 384	−872	+1429	− 151	− 427	−896
48	+1474	+ 57	− 33	−751	+1495	+ 33	− 5	−824	+1481	+ 12	− 22	−876
49	+1405	+ 194	+ 292	−651	+1438	+ 170	+ 338	−740	+1430	+ 164	+ 349	−813
50	+1256	+ 311	+ 561	−516	+1297	+ 292	+ 618	−620	+1288	+ 296	+ 661	−706
51	+1048	+ 398	+ 741	−354	+1085	+ 392	+ 829	−467	+1078	+ 404	+ 886	−565
52	+ 820	+ 446	+ 825	−177	+ 836	+ 468	+ 930	−286	+ 829	+ 484	+1011	−395
53	+ 609	+ 452	+ 819	+ 4	+ 575	+ 511	+ 932	− 87	+ 564	+ 534	+1031	−205
54	+ 432	+ 414	+ 734	+177	+ 342	+ 511	+ 839	+116	+ 313	+ 551	+ 955	− 3
55	+ 308	+ 342	+ 600	+330	+ 159	+ 468	+ 672	+313	+ 97	+ 534	+ 790	+206
56	+ 237	+ 245	+ 451	+463	+ 51	+ 385	+ 465	+490	− 55	+ 479	+ 556	+411
57	+ 219	+ 127	+ 292	+569	+ 26	+ 264	+ 246	+629	− 125	+ 384	+ 297	+597
58	+ 247	− 6	+ 161	+640	+ 66	+ 119	+ 72	+726	− 105	+ 249	+ 66	+742
59	+ 312	− 148	+ 89	+665	+ 151	− 38	− 26	+768	− 15	+ 87	− 89	+827

TABLE III.—*Continued.*

MUTUAL PERIODIC PERTURBATIONS OF VENUS AND THE EARTH.

The term of long period is omitted. The tabular unit is 0″.01 in δu and $\delta v'$, and 10^{-8} in $\delta\rho$ and $\delta\rho'$.

i	System 6.				System 7.				System 8.			
	δu	$\delta v'$	$\delta\rho$	$\delta\rho'$	δu	$\delta v'$	$\delta\rho$	$\delta\rho'$	δu	$\delta v'$	$\delta\rho$	$\delta\rho'$
0	− 57	+ 45	− 130	+838	− 225	+ 170	− 33	+759	− 422	+ 319	+ 73	+655
1	+ 73	− 129	− 94	+826	− 110	+ 5	− 39	+773	− 316	+ 156	+ 59	+682
2	+ 166	− 292	+ 58	+741	− 12	− 155	+ 56	+726	− 226	− 3	+ 134	+654
3	+ 186	− 426	+ 296	+596	+ 38	− 301	+ 231	+629	− 177	− 148	+ 272	+581
4	+ 113	− 519	+ 558	+406	+ 21	− 422	+ 446	+487	− 179	− 273	+ 434	+471
5	− 55	− 565	+ 794	+195	− 82	− 509	+ 663	+312	− 238	− 375	+ 596	+334
6	− 294	− 565	+ 961	− 23	− 266	− 552	+ 840	+115	− 353	− 449	+ 733	+178
7	− 573	− 526	+1039	−232	− 511	− 547	+ 943	− 94	− 522	− 490	+ 827	+ 5
8	− 860	− 453	+1017	−421	− 790	− 493	+ 950	−298	− 735	− 490	+ 845	−175
9	−1128	− 351	+ 892	−588	−1066	− 400	+ 855	−483	− 973	− 442	+ 776	−349
10	−1349	− 228	+ 668	−723	−1302	− 275	+ 651	−639	−1201	− 348	+ 605	−518
11	−1496	− 88	+ 361	−823	−1470	− 131	+ 368	−758	−1378	− 217	+ 343	−659
12	−1549	+ 64	− 9	−879	−1549	+ 23	+ 23	−836	−1480	− 62	+ 13	−762
13	−1494	+ 223	− 405	−893	−1524	+ 182	− 352	−876	−1481	+ 107	− 350	−828
14	−1335	+ 381	− 790	−860	−1390	+ 339	− 732	−875	−1378	+ 277	− 714	−850
15	−1089	+ 533	−1125	−788	−1150	+ 489	−1076	−834	−1178	+ 440	−1048	−831
16	− 775	+ 671	−1382	−682	− 815	+ 624	−1355	−757	− 875	+ 588	−1327	−774
17	− 427	+ 796	−1541	−547	− 419	+ 742	−1534	−640	− 500	+ 718	−1521	−680
18	− 73	+ 902	−1592	−396	+ 2	+ 833	−1595	−496	− 75	+ 821	−1606	−555
19	+ 266	+ 987	−1528	−231	+ 409	+ 901	−1538	−328	+ 370	+ 895	−1567	−406
20	+ 563	+1046	−1357	− 59	+ 759	+ 944	−1364	−151	+ 782	+ 934	−1402	−238
21	+ 799	+1073	−1083	+114	+1030	+ 966	−1098	+ 29	+1124	+ 941	−1126	− 58
22	+ 957	+1062	− 731	+277	+1205	+ 967	− 754	+203	+1362	+ 920	− 766	+123
23	+1023	+1009	− 349	+425	+1274	+ 943	− 359	+366	+1479	+ 876	− 356	+298
24	+1001	+ 919	+ 47	+544	+1232	+ 890	+ 40	+512	+1467	+ 812	+ 70	+459
25	+ 899	+ 801	+ 412	+632	+1087	+ 809	+ 461	+634	+1333	+ 735	+ 486	+599
26	+ 737	+ 663	+ 726	+687	+ 842	+ 696	+ 816	+726	+1083	+ 641	+ 861	+713
27	+ 540	+ 517	+ 962	+716	+ 541	+ 565	+1090	+784	+ 742	+ 531	+1165	+795
28	+ 318	+ 366	+1142	+729	+ 207	+ 419	+1266	+809	+ 340	+ 406	+1368	+846
29	+ 90	+ 213	+1251	+728	− 121	+ 270	+1339	+802	− 91	+ 268	+1457	+861
30	− 139	+ 57	+1282	+721	− 422	+ 121	+1321	+772	− 498	+ 126	+1428	+843
31	− 359	− 102	+1236	+705	− 677	− 23	+1230	+727	− 849	− 11	+1294	+796
32	− 562	− 262	+1118	+679	− 884	− 164	+1071	+673	−1126	− 140	+1084	+726
33	− 731	− 422	+ 926	+645	−1037	− 306	+ 855	+615	−1312	− 260	+ 813	+642
34	− 852	− 574	+ 675	+599	−1128	− 445	+ 589	+553	−1406	− 370	+ 510	+548
35	− 923	− 718	+ 374	+545	−1153	− 579	+ 284	+487	−1418	− 474	+ 187	+453
36	− 935	− 852	+ 39	+480	−1108	− 709	− 40	+414	−1346	− 577	− 136	+357
37	− 886	− 972	− 315	+399	− 991	− 828	− 365	+334	−1202	− 676	− 449	+265
38	− 773	−1071	− 683	+300	− 803	− 933	− 688	+246	− 982	− 767	− 743	+177
39	− 589	−1143	−1024	+178	− 558	−1024	− 983	+150	− 697	− 850	− 997	+ 88
40	− 335	−1177	−1317	+ 33	− 262	−1093	−1240	+ 45	− 362	− 920	−1203	− 3
41	− 28	−1165	−1526	−130	+ 79	−1139	−1436	− 72	+ 12	− 977	−1354	− 96
42	+ 313	−1100	−1622	−297	+ 450	−1147	−1550	−201	+ 407	−1017	−1438	−192
43	+ 650	− 989	−1599	−458	+ 825	−1112	−1556	−337	+ 808	−1036	−1454	−291
44	+ 959	− 837	−1456	−600	+1172	−1024	−1451	−473	+1190	−1024	−1381	−393
45	+1215	− 658	−1207	−717	+1462	− 887	−1233	−599	+1535	− 974	−1213	−491
46	+1400	− 466	− 881	−801	+1660	− 709	− 930	−703	+1805	− 878	− 961	−581
47	+1505	− 269	− 496	−850	+1753	− 502	− 564	−773	+1974	− 737	− 629	−655
48	+1524	− 77	− 87	−863	+1738	− 285	− 167	−807	+2021	− 555	− 248	−705
49	+1454	+ 106	+ 307	−837	+1629	− 67	+ 222	−803	+1941	− 342	+ 135	−725
50	+1298	+ 270	+ 651	−769	+1433	+ 137	+ 576	−758	+1741	− 125	+ 488	−706
51	+1070	+ 407	+ 909	−653	+1168	+ 319	+ 866	−675	+1450	+ 91	+ 785	−650
52	+ 802	+ 513	+1062	−500	+ 861	+ 471	+1063	−556	+1100	+ 287	+ 998	−555
53	+ 521	+ 582	+1101	−316	+ 534	+ 586	+1151	−403	+ 720	+ 454	+1118	−425
54	+ 256	+ 610	+1045	−117	+ 225	+ 654	+1129	−225	+ 346	+ 580	+1138	−271
55	+ 34	+ 599	+ 890	+ 93	− 43	+ 677	+ 988	− 27	+ 7	+ 662	+1051	− 97
56	− 135	+ 553	+ 671	+299	− 239	+ 651	+ 773	+180	− 266	+ 693	+ 875	+ 90
57	− 224	+ 472	+ 414	+491	− 348	+ 579	+ 520	+379	− 446	+ 670	+ 639	+273
58	− 237	+ 356	+ 163	+656	− 371	+ 467	+ 272	+552	− 522	+ 592	+ 392	+440
59	− 174	+ 212	− 33	+776	− 322	+ 328	+ 76	+683	− 504	+ 471	+ 188	+572

TABLE III.—*Concluded.*

MUTUAL PERIODIC PERTURBATIONS OF VENUS AND THE EARTH.

The term of long period is omitted. The tabular unit is $0''.01$ in δu and $\delta v'$, and 10^{-8} in $\delta\rho$ and $\delta\rho'$.

i	System 9.				System 10.				System 11.			
	δu	$\delta v'$	$\delta\rho$	$\delta\rho'$	δu	$\delta v'$	$\delta\rho$	$\delta\rho'$	δu	$\delta v'$	$\delta\rho$	$\delta\rho'$
0	− 598	+447	+ 196	+563	− 548	+407	+ 300	+495	− 205	+158	+ 341	+471
1	− 544	+318	+ 174	+583	− 616	+355	+ 295	+506	− 371	+169	+ 372	+454
2	− 482	+176	+ 237	+552	− 660	+285	+ 355	+469	− 540	+175	+ 468	+393
3	− 444	+ 36	+ 357	+487	− 698	+200	+ 465	+398	− 718	+179	+ 591	+306
4	− 446	− 90	+ 493	+394	− 747	+109	+ 579	+301	− 897	+176	+ 700	+207
5	− 493	−196	+ 613	+285	− 811	+ 19	+ 673	+193	−1066	+165	+ 767	+103
6	− 576	−278	+ 699	+162	− 889	− 62	+ 727	+ 82	−1208	+141	+ 783	+ 0
7	− 690	−338	+ 744	+ 26	− 980	−127	+ 732	− 32	−1321	+109	+ 757	−104
8	− 828	−370	+ 736	−121	−1073	−171	+ 681	−142	−1409	+ 77	+ 662	−205
9	− 981	−371	+ 667	−272	−1164	−194	+ 577	−254	−1470	+ 51	+ 516	−300
10	−1138	−335	+ 522	−420	−1249	−193	+ 416	−364	−1497	+ 39	+ 342	−384
11	−1274	−260	+ 298	−553	−1290	−166	+ 203	−472	−1484	+ 41	+ 114	−456
12	−1360	−147	− 3	−661	−1310	−113	− 60	−569	−1423	+ 59	− 147	−517
13	−1372	− 1	− 351	−740	−1274	− 29	− 364	−648	−1314	+ 95	− 427	−566
14	−1288	+164	− 715	−783	−1176	+ 83	− 693	−699	−1154	+147	− 711	−603
15	−1106	+339	−1055	−786	−1000	+222	−1015	−716	− 938	+220	− 987	−624
16	− 835	+504	−1328	−753	− 741	+374	−1301	−695	− 669	+311	−1237	−620
17	− 493	+654	−1520	−682	− 415	+530	−1511	−637	− 346	+420	−1438	−583
18	− 106	+779	−1609	−576	− 40	+671	−1617	−549	+ 19	+536	−1561	−510
19	+ 308	+876	−1582	−440	+ 353	+791	−1605	−431	+ 413	+651	−1579	−403
20	+ 710	+938	−1438	−282	+ 726	+878	−1469	−293	+ 793	+749	−1479	−270
21	+1072	+964	−1180	−108	+1049	+933	−1225	−135	+1126	+821	−1260	−116
22	+1359	+948	− 820	+ 68	+1323	+949	− 880	+ 33	+1387	+862	− 940	+ 40
23	+1532	+899	− 399	+242	+1501	+927	− 467	+207	+1547	+867	− 544	+202
24	+1580	+816	+ 54	+405	+1572	+863	− 13	+376	+1600	+835	− 101	+359
25	+1493	+711	+ 494	+551	+1525	+765	+ 446	+530	+1546	+769	+ 355	+509
26	+1278	+592	+ 885	+677	+1353	+632	+ 872	+659	+1379	+667	+ 791	+644
27	+ 957	+466	+1210	+778	+1070	+479	+1221	+759	+1116	+533	+1174	+757
28	+ 558	+336	+1436	+849	+ 701	+316	+1471	+833	+ 766	+372	+1460	+841
29	+ 106	+205	+1545	+884	+ 271	+153	+1600	+874	+ 353	+192	+1627	+887
30	− 355	+ 73	+1534	+883	− 179	− 1	+1609	+886	− 84	+ 6	+1659	+896
31	− 791	− 59	+1404	+848	− 621	−142	+1501	+867	− 515	−171	+1565	+870
32	−1164	−183	+1169	+780	−1026	−270	+1278	+815	− 909	−330	+1357	+814
33	−1439	−297	+ 861	+690	−1358	−383	+ 961	+733	−1234	−466	+1050	+735
34	−1594	−393	+ 497	+581	−1592	−477	+ 574	+621	−1476	−573	+ 673	+632
35	−1633	−471	+ 134	+465	−1704	−550	+ 154	+491	−1629	−652	+ 250	+511
36	−1560	−537	− 225	+345	−1684	−602	− 256	+351	−1661	−707	− 183	+370
37	−1402	−594	− 544	+226	−1549	−630	− 620	+209	−1579	−733	− 596	+216
38	−1173	−645	− 818	+115	−1317	−639	− 923	+ 77	−1384	−730	− 959	+ 60
39	− 879	−694	−1058	+ 11	−1017	−637	−1149	− 43	−1098	−703	−1232	− 89
40	− 540	−740	−1237	− 84	− 674	−631	−1301	−150	− 752	−654	−1403	−219
41	− 159	−783	−1347	−169	− 303	−627	−1384	−244	− 375	−595	−1473	−324
42	+ 244	−813	−1390	−245	+ 80	−626	−1396	−324	+ 3	−532	−1450	−403
43	+ 651	−838	−1365	−315	+ 465	−626	−1342	−391	+ 364	−477	−1357	−462
44	+1039	−849	−1275	−383	+ 835	−623	−1225	−442	+ 697	−432	−1208	−504
45	+1399	−845	−1121	−449	+1181	−617	−1049	−478	+1001	−396	−1009	−531
46	+1708	−821	− 911	−509	+1482	−607	− 832	−502	+1264	−368	− 778	−542
47	+1946	−768	− 642	−562	+1728	−591	− 580	−517	+1480	−346	− 520	−536
48	+2089	−681	− 322	−601	+1908	−566	− 303	−526	+1640	−328	− 248	−515
49	+2122	−555	+ 27	−621	+2014	−524	− 15	−527	+1743	−312	+ 17	−481
50	+2026	−393	+ 372	−611	+2027	−460	+ 276	−514	+1784	−298	+ 265	−434
51	+1803	−207	+ 678	−570	+1939	−371	+ 548	−479	+1761	−283	+ 487	−378
52	+1482	− 11	+ 912	−499	+1744	−254	+ 779	−418	+1679	−259	+ 673	−314
53	+1093	+178	+1054	−396	+1454	−116	+ 942	−327	+1528	−221	+ 813	−236
54	+ 679	+344	+1097	−269	+1097	+ 27	+1017	−213	+1313	−169	+ 895	−145
55	+ 279	+477	+1042	−117	+ 709	+166	+ 995	− 80	+1044	−101	+ 900	− 31
56	− 71	+569	+ 906	+ 49	+ 331	+284	+ 884	+ 66	+ 743	− 27	+ 825	+100
57	− 344	+612	+ 710	+217	+ 3	+371	− 716	+209	+ 451	+ 43	+ 686	+234
58	− 523	+603	+ 497	+370	− 254	+418	+ 536	+337	+ 189	+ 99	+ 525	+354
59	− 601	+548	+ 312	+489	− 434	+430	+ 385	+437	− 27	+137	+ 397	+437

TABLE IV*a*.

PERTURBATIONS OF THE G-COÖRDINATE X OF VENUS.

The tabular unit is 10^{-8}.

System i	0	1	2	3	4	5	6	7	8	9	10	11
0	+ 47	+ 57	+ 92	+137	+181	+217	+218	+184	+142	+107	+ 71	+ 52
1	+ 37	+ 50	+ 81	+118	+159	+197	+212	+183	+140	+ 92	+ 52	+ 30
2	− 11	+ 11	+ 41	+ 71	+109	+149	+176	+158	+114	+ 59	+ 8	− 23
3	− 92	− 59	− 24	+ 5	+ 39	+ 75	+112	+111	+ 72	+ 9	− 52	− 97
4	−193	−154	−111	− 79	− 50	− 14	+ 26	+ 46	+ 14	− 48	−122	−183
5	−305	−269	−217	−183	−156	−122	− 79	− 41	− 55	−111	−191	−268
6	−412	−396	−338	−290	−267	−237	−197	−146	−136	−179	−261	−348
7	−506	−519	−465	−405	−380	−354	−318	−263	−230	−251	−326	−421
8	−581	−628	−586	−519	−487	−468	−437	−384	−331	−330	−386	−482
9	−636	−715	−696	−629	−583	−569	−546	−502	−437	−410	−443	−530
10	−668	−763	−780	−718	−661	−650	−636	−599	−539	−488	−494	−565
11	−674	−777	−827	−783	−718	−699	−697	−672	−619	−556	−534	−582
12	−660	−759	−828	−814	−745	−714	−722	−710	−669	−606	−561	−580
13	−621	−710	−790	−803	−745	−699	−707	−712	−682	−625	−569	−560
14	−559	−639	−713	−749	−709	−652	−652	−671	−659	−610	−552	−523
15	−479	−541	−614	−659	−640	−580	−564	−591	−596	−559	−507	−466
16	−383	−431	−495	−542	−540	−487	−452	−475	−496	−476	−432	−389
17	−279	−308	−364	−410	−417	−374	−325	−337	−368	−366	−331	−298
18	−171	−183	−231	−273	−283	−251	−195	−187	−221	−236	−213	−187
19	− 54	− 61	− 98	−137	−149	−123	− 68	− 38	− 66	− 95	− 84	− 64
20	+ 60	+ 56	+ 25	− 13	− 26	− 1	+ 48	+ 93	+ 80	+ 46	+ 41	+ 58
21	+170	+160	+136	+100	+ 82	+106	+151	+203	+209	+179	+155	+171
22	+269	+253	+229	+197	+174	+192	+236	+288	+312	+291	+262	+266
23	+350	+330	+305	+275	+249	+259	+299	+350	+384	+377	+348	+342
24	+406	+392	+363	+333	+307	+306	+341	+387	+430	+434	+412	+398
25	+441	+436	+407	+373	+347	+338	+367	+412	+451	+466	+456	+437
26	+461	+460	+435	+400	+374	+361	+378	+418	+457	+475	+476	+464
27	+468	+471	+451	+417	+388	+372	+380	+418	+454	+478	+483	+478
28	+476	+475	+459	+426	+394	+381	+379	+411	+447	+472	+481	+484
29	+480	+475	+463	+432	+397	+381	+379	+405	+440	+465	+475	+485
30	+484	+479	+467	+438	+401	+380	+380	+401	+434	+459	+473	+479
31	+490	+484	+476	+445	+406	+380	+380	+397	+431	+457	+470	+476
32	+491	+492	+483	+455	+414	+380	+379	+392	+426	+456	+469	+473
33	+485	+497	+488	+460	+417	+379	+375	+387	+419	+451	+467	+466
34	+468	+492	+489	+463	+420	+377	+362	+375	+403	+437	+456	+456
35	+439	+468	+479	+458	+414	+366	+341	+354	+377	+411	+432	+436
36	+397	+426	+448	+437	+392	+342	+310	+314	+338	+368	+391	+399
37	+340	+360	+394	+396	+353	+301	+262	+260	+282	+311	+331	+343
38	+263	+270	+309	+327	+293	+240	+196	+184	+208	+236	+255	+264
39	+167	+166	+199	+230	+211	+155	+109	+ 90	+112	+141	+161	+166
40	+ 56	+ 47	+ 69	+107	+104	+ 51	+ 2	− 20	− 4	+ 33	+ 55	+ 57
41	− 67	− 80	− 74	− 38	− 22	− 67	−121	−145	−127	− 90	− 59	− 58
42	−188	−206	−215	−188	−161	−192	−254	−283	−265	−222	−180	−172
43	−301	−325	−344	−336	−306	−319	−381	−420	−407	−357	−304	−282
44	−397	−430	−459	−464	−443	−438	−494	−549	−543	−486	−423	−385
45	−473	−510	−545	−568	−557	−544	−589	−656	−665	−609	−534	−475
46	−528	−560	−605	−634	−638	−626	−655	−729	−760	−712	−628	−555
47	−566	−578	−626	−665	−681	−678	−694	−764	−818	−791	−702	−617
48	−585	−574	−611	−660	−685	−692	−703	−759	−833	−833	−749	−657
49	−588	−547	−566	−619	−653	−668	−679	−723	−805	−838	−775	−675
50	−572	−508	−506	−549	−588	−609	−627	−659	−737	−796	−766	−668
51	−539	−457	−431	−454	−499	−525	−545	−571	−639	−716	−723	−639
52	−491	−401	−347	−350	−391	−423	−447	−470	−525	−605	−644	−589
53	−431	−339	−268	−248	−276	−311	−333	−357	−401	−477	−535	−518
54	−362	−274	−193	−152	−162	−197	−222	−247	−281	−346	−414	−428
	[illegible]	[illegible]	−122	− 69	− 58	− 88	−111	−137	−167	−219	−286	−324
			[illegible]	+ 2	+ 31	+ 14	− 14	− 37	− 64	−106	−164	−211
				+ 60	+101	+102	+ 73	+ 48	+ 20	− 14	− 60	−105
				+104	+151	+167	+144	+115	+ 85	+ 54	+ 16	− 18
				[illegible]	+179	+208	+194	+161	+128	+ 94	+ 59	+ 36

TABLE IV*b*.

PERTURBATIONS OF THE G-COÖRDINATE Y OF VENUS.

The tabular unit is 10^{-8}.

System *i*	0	1	2	3	4	5	6	7	8	9	10	11
0	−164	−370	−379	−274	−176	− 78	+ 42	+160	+300	+423	+385	+146
1	−100	−357	−436	−365	−275	−199	− 87	+ 39	+181	+337	+377	+202
2	− 22	−310	−454	−430	−350	−290	−203	− 77	+ 68	+239	+348	+251
3	+ 70	−224	−424	−458	−394	−345	−287	−175	− 30	+145	+304	+294
4	+172	−111	−350	−438	−405	−360	−330	−246	−108	+ 65	+252	+324
5	+273	+ 22	−236	−373	−378	−341	−328	−281	−162	+ 3	+200	+339
6	+355	+164	− 97	−272	−318	−294	−289	−277	−190	− 38	+154	+334
7	+410	+298	+ 53	−148	−231	−228	−225	−239	−193	− 65	+115	+313
8	+435	+406	+202	− 11	−124	−146	−144	−171	−167	− 72	+ 89	+286
9	+436	+481	+340	+131	− 6	− 54	− 59	− 87	−114	− 62	+ 73	+258
10	+415	+523	+452	+267	+114	+ 46	+ 32	+ 7	− 38	− 31	+ 70	+232
11	+389	+531	+533	+390	+230	+146	+120	+ 98	+ 49	+ 19	+ 78	+214
12	+361	+519	+583	+490	+339	+244	+205	+185	+139	+ 86	+101	+205
13	+339	+495	+602	+571	+442	+338	+290	+267	+227	+165	+142	+207
14	+331	+472	+600	+626	+533	+429	+372	+348	+315	+254	+201	+229
15	+339	+457	+590	+660	+610	+515	+454	+427	+399	+348	+279	+268
16	+363	+457	+579	+673	+673	+597	+536	+508	+484	+440	+371	+329
17	+406	+472	+575	+679	+721	+673	+615	+588	+568	+534	+473	+409
18	+469	+502	+582	+678	+749	+738	+690	+667	+650	+624	+579	+505
19	+550	+548	+602	+680	+760	+787	+757	+735	+729	+710	+675	+610
20	+639	+606	+629	+685	+759	+812	+809	+792	+793	+786	+761	+710
21	+729	+668	+660	+692	+747	+814	+837	+832	+837	+845	+829	+795
22	+804	+729	+691	+698	+729	+788	+835	+847	+858	+877	+875	+855
23	+848	+779	+715	+696	+704	+743	+799	+830	+846	+875	+888	+880
24	+854	+801	+725	+681	+671	+681	+729	+778	+801	+833	+861	+864
25	+812	+786	+717	+653	+621	+606	+630	+683	+717	+752	+792	+809
26	+723	+722	+676	+602	+552	+517	+510	+551	+594	+632	+676	+710
27	+594	+609	+593	+531	+465	+416	+383	+397	+441	+478	+522	+567
28	+428	+455	+467	+428	+360	+297	+248	+228	+262	+301	+341	+390
29	+237	+269	+301	+295	+239	+174	+113	+ 66	+ 70	+106	+138	+187
30	+ 26	+ 67	+107	+132	+100	+ 40	− 22	− 90	−116	− 92	− 65	− 27
31	−187	−142	−099	− 54	− 51	− 98	−155	−227	−280	−283	−261	−235
32	−388	−340	−300	−250	−216	−237	−284	−346	−416	−449	−440	−424
33	−565	−523	−484	−439	−389	−375	−403	−451	−520	−580	−589	−582
34	−702	−673	−637	−602	−547	−505	−509	−538	−594	−667	−702	−703
35	−796	−785	−754	−729	−685	−625	−601	−609	−644	−711	−770	−787
36	−846	−852	−835	−814	−789	−728	−679	−664	−675	−723	−788	−829
37	−857	−875	−872	−860	−850	−807	−743	−704	−691	−712	−774	−830
38	−832	−857	−869	−867	−869	−850	−791	−729	−695	−688	−729	−794
39	−779	−810	−833	−840	−850	−858	−820	−749	−690	−659	−669	−729
40	−704	−741	−769	−787	−802	−835	−827	−761	−685	−631	−608	−647
41	−613	−659	−690	−713	−736	−778	−810	−767	−682	−607	−554	−561
42	−518	−568	−601	−629	−655	−704	−766	−763	−680	−589	−512	−483
43	−428	−475	−513	−539	−568	−618	−702	−743	−686	−582	−484	−421
44	−350	−383	−426	−452	−478	−528	−622	−705	−689	−586	−468	−379
45	−289	−299	−341	−369	−393	−435	−530	−645	−681	−597	−469	−353
46	−249	−226	−260	−291	−310	−344	−432	−569	−659	−612	−480	−345
47	−227	−168	−183	−214	−229	−254	−329	−472	−610	−618	−499	−352
48	−220	−126	−111	−139	−152	−166	−223	−360	−532	−606	−521	−363
49	−227	−103	− 47	− 64	− 77	− 79	−116	−235	−425	−566	−535	−387
50	−240	− 94	− 3	+ 9	− 2	+ 8	− 10	−104	−295	−488	−530	−410
51	−258	− 96	+ 25	+ 75	+ 74	+ 88	+ 97	+ 30	−148	−377	−497	−424
52	−278	−105	+ 40	+124	+143	+163	+192	+160	+ 8	−235	−427	−427
53	−297	−124	+ 35	+151	+200	+226	+270	+276	+161	− 77	−324	−407
54	−314	−150	+ 13	+151	+234	+272	+327	+368	+299	+ 84	−194	−359
55	−319	−184	− 21	+124	+237	+298	+358	+428	+409	+235	− 49	−285
56	−311	−226	− 68	+ 74	+205	+291	+360	+449	+483	+361	+ 92	−192
57	−290	−273	−132	+ 8	+136	+247	+329	+427	+509	+448	+215	− 94
58	−257	−319	−212	− 77	+ 42	+164	+264	+365	+481	+487	+305	+ 1
59	−215	−355	−298	−173	− 66	+ 50	+165	+274	+407	+477	+361	+ 81

TABLE V.

RECTANGULAR G-COÖRDINATES X AND Y OF VENUS, REFERRED, IN EACH SYSTEM, TO AN AXIS OF X PASSING THROUGH THE MEAN SUN.

i	System 0.		System 1.		System 2.	
	X	Y	X	Y	X	Y
0	+0.264 4388	+0.005 3159	+0.267 8642	+0.017 2810	+0.273 5960	+0.024 1368
1	+0.268 2411	−0.070 6564	+0.271 9461	−0.058 8025	+0.277 8772	−0.051 8927
2	+0.280 0227	−0.145 8383	+0.284 0031	−0.134 0369	+0.290 1029	−0.127 0162
3	+0.299 6585	−0.219 3893	+0.303 8978	−0.207 5810	+0.310 1265	−0.200 3988
4	+0.326 9346	−0.290 4863	+0.331 4038	−0.278 6154	+0.337 7133	−0.271 2285
5	+0.361 5487	−0.358 3344	+0.366 2082	−0.346 3496	+0.372 5452	−0.338 7248
6	+0.403 1159	−0.422 1763	+0.407 9159	−0.410 0328	+0.414 2234	−0.402 1483
7	+0.451 1712	−0.481 2993	+0.456 0557	−0.468 9619	+0.462 2754	−0.460 8077
8	+0.505 1754	−0.535 0445	+0.510 0841	−0.522 4898	+0.516 1594	−0.514 0671
9	+0.564 5219	−0.582 8139	+0.569 3924	−0.570 0313	+0.575 2710	−0.561 3528
10	+0.628 5438	−0.624 0792	+0.633 3160	−0.611 0692	+0.638 9518	−0.602 1602
11	+0.696 5229	−0.658 3852	+0.701 1382	−0.645 1617	+0.706 4955	−0.636 0564
12	+0.767 6965	−0.685 3576	+0.772 1023	−0.671 9440	+0.777 1566	−0.662 6856
13	+0.841 2692	−0.704 7061	+0.845 4198	−0.691 1353	+0.850 1572	−0.681 7730
14	+0.916 4199	−0.716 2270	+0.920 2784	−0.702 5389	+0.924 6978	−0.693 1251
15	+0.992 3125	−0.719 8055	+0.995 8539	−0.706 0456	+0.999 9648	−0.696 6325
16	+1.068 1057	−0.715 4169	+1.071 3157	−0.701 6335	+1.075 1404	−0.692 2716
17	+1.142 9619	−0.703 1247	+1.145 8394	−0.689 3674	+1.149 4106	−0.680 1025
18	+1.216 0577	−0.683 0801	+1.218 6136	−0.669 3977	+1.221 9735	−0.660 2691
19	+1.286 5933	−0.655 5200	+1.288 8493	−0.641 9577	+1.292 0488	−0.632 9963
20	+1.353 7991	−0.620 7640	+1.355 7887	−0.607 3599	+1.358 8839	−0.598 5883
21	+1.416 9464	−0.579 2086	+1.418 7118	−0.565 9943	+1.421 7633	−0.557 4240
22	+1.475 3529	−0.531 3238	+1.476 9448	−0.518 3206	+1.480 0138	−0.509 9523
23	+1.528 3911	−0.477 6470	+1.529 8660	−0.464 8651	+1.533 0133	−0.456 6884
24	+1.575 4942	−0.418 7748	+1.576 9123	−0.406 2148	+1.580 1954	−0.398 2084
25	+1.616 1628	−0.355 3577	+1.617 5843	−0.343 0090	+1.621 0558	−0.335 1418
26	+1.649 9691	−0.288 0919	+1.651 4516	−0.275 9346	+1.655 1564	−0.268 1673
27	+1.676 5593	−0.217 7116	+1.678 1568	−0.205 7162	+1.682 1302	−0.198 0049
28	+1.695 6592	−0.144 9820	+1.697 4188	−0.133 1099	+1.701 6857	−0.125 4065
29	+1.707 0734	−0.070 6903	+1.709 0351	−0.058 8960	+1.713 6087	−0.051 1511
30	+1.710 6894	+0.004 3615	+1.712 8842	+0.016 1297	+1.717 7659	+0.023 9652
31	+1.706 4775	+0.079 3664	+1.708 9249	+0.091 1628	+1.714 1050	+0.099 1360
32	+1.694 4891	+0.153 5190	+1.697 1990	+0.165 3988	+1.702 6556	+0.173 5529
33	+1.674 8587	+0.226 0244	+1.677 8288	+0.238 0408	+1.683 5308	+0.246 4124
34	+1.647 8008	+0.296 1060	+1.651 0177	+0.308 3085	+1.656 9253	+0.316 9261
35	+1.613 6080	+0.363 0127	+1.617 0473	+0.375 4452	+1.623 1144	+0.384 3269
36	+1.572 6476	+0.426 0265	+1.576 2761	+0.438 7253	+1.582 4506	+0.447 8781
37	+1.525 3583	+0.484 4694	+1.529 1350	+0.497 4622	+1.535 3622	+0.506 8825
38	+1.472 2450	+0.537 7122	+1.476 1240	+0.551 0155	+1.482 3484	+0.560 6879
39	+1.413 8753	+0.585 1789	+1.417 8080	+0.598 7973	+1.423 9759	+0.608 6959
40	+1.350 8731	+0.626 3539	+1.354 8096	+0.640 2802	+1.360 8720	+0.650 3687
41	+1.283 9122	+0.660 7876	+1.287 8040	+0.675 0020	+1.293 7184	+0.685 2350
42	+1.213 7108	+0.688 0995	+1.217 5124	+0.702 5721	+1.223 2454	+0.712 8968
43	+1.141 0227	+0.707 9845	+1.144 6940	+0.722 6753	+1.150 2220	+0.733 0331
44	+1.066 6304	+0.720 2157	+1.070 1388	+0.735 0768	+1.075 4484	+0.745 4066
45	+0.991 3369	+0.724 6478	+0.994 6591	+0.739 6239	+0.999 7488	+0.749 8650
46	+0.915 9571	+0.721 2182	+0.919 0806	+0.736 2498	+0.923 9587	+0.746 3434
47	+0.841 3093	+0.709 9494	+0.844 2332	+0.724 9743	+0.848 9201	+0.734 8666
48	+0.768 2082	+0.690 9488	+0.770 9409	+0.705 9042	+0.775 4674	+0.715 5486
49	+0.697 4541	+0.664 4077	+0.700 0152	+0.679 2337	+0.704 4196	+0.688 5925
50	+0.629 8261	+0.630 6012	+0.632 2439	+0.645 2428	+0.636 5703	+0.654 2871
51	+0.566 0715	+0.589 8843	+0.568 3830	+0.604 2938	+0.572 6800	+0.613 0068
52	+0.506 8987	+0.542 6896	+0.509 1472	+0.556 8286	+0.513 4653	+0.565 2062
53	+0.452 9685	+0.489 5225	+0.455 2029	+0.503 3636	+0.459 5903	+0.511 4133
54	+0.404 8861	+0.430 9573	+0.407 1576	+0.444 4847	+0.411 6602	+0.452 2270
55	+0.363 1948	+0.367 6314	+0.365 5549	+0.380 8404	+0.370 2130	+0.388 3082
56	+0.328 3680	+0.300 2365	+0.330 8664	+0.313 1351	+0.335 7125	+0.320 3708
57	+0.300 8042	+0.229 5127	+0.303 4856	+0.242 1210	+0.308 5436	+0.249 1751
58	+0.280 8207	+0.156 2403	+0.283 7236	+0.168 5895	+0.289 0073	+0.175 5176
59	+0.268 6504	+0.081 2302	+0.271 8049	+0.093 3621	+0.277 3170	+0.100 2240

TABLE V.—*Continued.*

RECTANGULAR G-COÖRDINATES X AND Y OF VENUS, REFERRED, IN EACH SYSTEM, TO AN AXIS OF X PASSING THROUGH THE MEAN SUN.

i	System 3.		System 4.		System 5.	
	X	*Y*	*X*	*Y*	*X*	*Y*
0	+0.279 8926	+0.024 4020	+0.285 2603	+0.018 4830	+0.288 6591	+0.008 0878
1	+0.284 2769	−0.051 3907	+0.289 6702	−0.056 9777	+0.293 0191	−0.067 0880
2	+0.296 5601	−0.126 2430	+0.301 9298	−0.131 4966	+0.305 1838	−0.141 3542
3	+0.316 5918	−0.199 3308	+0.321 8895	−0.204 2627	+0.325 0091	−0.213 9093
4	+0.344 1354	−0.269 8527	+0.349 3160	−0.274 4873	+0.352 2710	−0.283 9728
5	+0.378 8721	−0.337 0406	+0.383 8974	−0.341 4123	+0.386 6655	−0.350 7937
6	+0.420 4078	−0.400 1668	+0.425 2457	−0.404 3190	+0.427 8152	−0.413 6573
7	+0.468 2736	−0.458 5510	+0.472 9020	−0.462 5340	+0.475 2721	−0.471 8919
8	+0.521 9346	−0.511 5674	+0.526 3418	−0.515 4368	+0.528 5222	−0.524 8756
9	+0.580 7952	−0.558 6507	+0.584 9803	−0.562 4651	+0.585 9911	−0.572 0427
10	+0.644 2076	−0.599 3022	+0.648 1798	−0.603 1207	+0.650 0504	−0.612 8887
11	+0.711 4758	−0.633 0942	+0.715 2549	−0.636 9740	+0.717 0241	−0.646 9765
12	+0.781 8659	−0.659 6737	+0.785 4819	−0.663 6672	+0.787 1933	−0.673 9389
13	+0.854 6132	−0.678 7647	+0.858 1036	−0.682 9185	+0.859 8058	−0.693 4842
14	+0.928 9294	−0.690 1730	+0.932 3397	−0.694 5257	+0.934 0825	−0.705 3977
15	+1.004 0113	−0.693 7852	+1.007 3925	−0.698 3667	+1.009 2254	−0.709 5459
16	+1.079 0492	−0.689 5714	+1.082 4565	−0.694 4006	+1.084 4261	−0.705 8759
17	+1.153 2351	−0.677 5823	+1.156 7246	−0.682 6691	+1.158 8744	−0.694 4182
18	+1.225 7719	−0.657 9526	+1.229 3983	−0.663 2956	+1.231 7646	−0.675 2864
19	+1.295 8808	−0.630 8962	+1.299 6944	−0.636 4839	+1.302 3064	−0.648 6759
20	+1.362 8093	−0.596 7058	+1.366 8546	−0.602 5160	+1.369 7311	−0.614 8635
21	+1.425 8394	−0.555 7489	+1.430 1522	−0.561 7502	+1.433 3011	−0.574 2026
22	+1.484 2931	−0.508 4647	+1.488 9006	−0.514 6175	+1.492 3178	−0.527 1222
23	+1.537 5412	−0.455 3593	+1.542 4596	−0.461 6180	+1.546 1296	−0.474 1214
24	+1.585 0084	−0.397 0007	+1.590 2425	−0.403 3157	+1.594 1387	−0.415 7661
25	+1.626 1801	−0.334 0124	+1.631 7225	−0.340 3342	+1.635 8083	−0.352 6827
26	+1.660 6072	−0.267 0689	+1.666 4388	−0.273 3482	+1.670 6690	−0.285 5523
27	+1.687 9113	−0.196 8867	+1.694 0012	−0.203 0779	+1.698 3230	−0.215 1035
28	+1.707 7882	−0.124 2202	+1.714 0957	−0.130 2819	+1.718 4514	−0.142 1049
29	+1.720 0122	−0.049 8508	+1.726 4883	−0.055 7493	+1.730 8162	−0.067 3551
30	+1.724 4388	+0.025 4188	+1.731 0274	+0.019 7085	+1.735 2660	+0.008 3225
31	+1.721 0059	+0.100 7745	+1.727 6465	+0.095 2669	+1.731 7363	+0.084 0928
32	+1.709 7359	+0.175 3989	+1.716 3660	+0.170 0960	+1.720 2518	+0.159 1154
33	+1.690 7351	+0.248 4792	+1.697 2920	+0.243 3714	+1.700 9265	+0.232 5558
34	+1.664 1950	+0.319 2167	+1.670 6195	+0.314 2840	+1.673 9638	+0.303 5951
35	+1.630 3893	+0.386 8336	+1.636 6271	+0.382 0454	+1.639 6529	+0.371 4384
36	+1.589 6721	+0.450 5830	+1.595 6761	+0.445 9010	+1.598 3674	+0.435 3259
37	+1.542 4753	+0.509 7560	+1.548 2079	+0.505 1363	+1.550 5612	+0.494 5411
38	+1.489 3032	+0.563 6915	+1.494 7388	+0.559 0849	+1.496 7627	+0.548 4197
39	+1.430 7300	+0.611 7835	+1.435 8543	+0.607 1386	+1.437 5701	+0.596 3565
40	+1.367 3923	+0.653 4880	+1.372 2027	+0.648 7530	+1.373 6443	+0.637 8131
41	+1.299 9821	+0.688 3306	+1.304 4890	+0.683 4552	+1.305 6998	+0.672 3257
42	+1.229 2412	+0.715 9113	+1.233 4657	+0.710 8508	+1.234 4980	+0.699 5075
43	+1.155 9507	+0.735 9114	+1.159 9243	+0.730 6274	+1.160 8358	+0.719 0571
44	+1.080 9236	+0.748 0961	+1.084 6868	+0.742 5595	+1.085 5389	+0.730 7598
45	+1.004 9938	+0.752 3188	+1.008 5958	+0.746 5108	+1.009 4492	+0.734 4915
46	+0.929 0085	+0.748 5233	+0.932 5029	+0.742 4365	+0.933 4172	+0.730 2183
47	+0.853 8157	+0.736 7446	+0.857 2595	+0.730 3837	+0.858 2903	+0.717 9976
48	+0.780 2560	+0.717 1082	+0.783 7064	+0.710 4897	+0.784 9033	+0.697 9763
49	+0.709 1529	+0.689 8295	+0.712 6639	+0.682 9817	+0.714 0686	+0.670 3895
50	+0.641 3017	+0.655 2115	+0.644 9229	+0.648 1733	+0.646 5668	+0.635 5557
51	+0.577 4611	+0.613 6402	+0.581 2344	+0.606 4605	+0.583 1378	+0.593 8734
52	+0.518 3431	+0.565 5804	+0.522 3027	+0.558 3153	+0.524 4736	+0.545 8162
53	+0.464 6055	+0.511 5714	+0.468 7755	+0.504 2817	+0.471 2105	+0.491 9264
54	+0.416 8456	+0.452 2194	+0.421 2379	+0.444 9681	+0.423 9224	+0.432 8084
55	+0.375 5907	+0.388 1916	+0.380 2062	+0.381 0400	+0.383 1147	+0.369 1218
56	+0.341 2944	+0.320 2069	+0.346 1218	+0.313 2132	+0.349 2206	+0.301 5727
57	+0.314 3306	+0.249 0281	+0.319 3477	+0.242 2440	+0.322 5952	+0.230 9071
58	+0.294 9887	+0.175 4508	+0.300 1638	+0.168 9220	+0.303 5124	+0.157 9020
59	+0.283 4712	+0.100 2967	+0.288 7653	+0.094 0597	+0.292 1646	+0.083 3577

TABLE V.—*Continued.*

RECTANGULAR G-COÖRDINATES X AND Y OF VENUS, REFERRED, IN EACH SYSTEM, TO AN AXIS OF X PASSING THROUGH THE MEAN SUN.

i	System 6.		System 7.		System 8.	
	X	*Y*	*X*	*Y*	*X*	*Y*
0	+0.289 3812	−0.004 3547	+0.287 0415	−0.015 9849	+0.281 8724	−0.023 8026
1	+0.293 5931	−0.079 3938	+0.291 0012	−0.091 0510	+0.285 5340	−0.099 0059
2	+0.305 5780	−0.153 5781	+0.302 7254	−0.165 3184	+0.296 9815	−0.173 4542
3	+0.325 2014	−0.226 1096	+0.322 0914	−0.237 9885	+0.316 1026	−0.246 3419
4	+0.352 2490	−0.296 2092	+0.348 8955	−0.308 2783	+0.342 7024	−0.316 8787
5	+0.386 4284	−0.363 1251	+0.382 8548	−0.375 4297	+0.376 5054	−0.384 2966
6	+0.427 3728	−0.426 1400	+0.423 6113	−0.438 7161	+0.417 1580	−0.447 8587
7	+0.474 6443	−0.484 5776	+0.470 7341	−0.497 4515	+0.464 2316	−0.506 8671
8	+0.527 7380	−0.537 8098	+0.523 7240	−0.550 9962	+0.517 2268	−0.560 6696
9	+0.586 0872	−0.585 2627	+0.582 0178	−0.598 7649	+0.575 5777	−0.608 6687
10	+0.649 0693	−0.626 4213	+0.644 9946	−0.640 2315	+0.638 6584	−0.650 3275
11	+0.716 0118	−0.660 8369	+0.711 9802	−0.674 9360	+0.705 7897	−0.685 1772
12	+0.786 1986	−0.688 1308	+0.782 2554	−0.702 4883	+0.776 2436	−0.712 8211
13	+0.858 8769	−0.707 9983	+0.855 0617	−0.722 5740	+0.849 2528	−0.732 9398
14	+0.933 2651	−0.720 2135	+0.929 6105	−0.734 9582	+0.924 0172	−0.745 2958
15	+1.008 5598	−0.724 6313	+1.005 0902	−0.739 4890	+0.999 7145	−0.749 7373
16	+1.083 9453	−0.721 1895	+1.080 6753	−0.736 0995	+1.075 5086	−0.746 1991
17	+1.158 6020	−0.709 9106	+1.155 5347	−0.724 8102	+1.150 5579	−0.734 7062
18	+1.231 7142	−0.690 9018	+1.228 8424	−0.705 7280	+1.224 0268	−0.715 3728
19	+1.302 4807	−0.664 3547	+1.299 7857	−0.679 0484	+1.295 0946	−0.688 4024
20	+1.370 1219	−0.630 5443	+1.367 5740	−0.645 0509	+1.362 9645	−0.654 0862
21	+1.433 8901	−0.589 8264	+1.431 4510	−0.604 0981	+1.426 8752	−0.612 7987
22	+1.493 0762	−0.542 6339	+1.490 7009	−0.556 6323	+1.486 1086	−0.564 9945
23	+1.547 0187	−0.489 4723	+1.544 6576	−0.503 1709	+1.539 9987	−0.511 2033
24	+1.595 1122	−0.430 9158	+1.592 7131	−0.444 2998	+1.587 9409	−0.452 0237
25	+1.636 8140	−0.367 6003	+1.634 3252	−0.380 6684	+1.629 3973	−0.388 1167
26	+1.671 6503	−0.300 2166	+1.669 0214	−0.312 9805	+1.663 9048	−0.320 1963
27	+1.699 2228	−0.229 5034	+1.696 4089	−0.241 9865	+1.691 0789	−0.249 0223
28	+1.719 2138	−0.156 2412	+1.716 1761	−0.168 4765	+1.710 6192	−0.175 3907
29	+1.731 3898	−0.081 2404	+1.728 0986	−0.093 2689	+1.722 3122	−0.100 1249
30	+1.735 6043	−0.005 3345	+1.732 0410	−0.017 2049	+1.726 0340	−0.024 0644
31	+1.731 8010	+0.070 6309	+1.727 9580	+0.058 8662	+1.721 7514	+0.051 9430
32	+1.720 0145	+0.145 8077	+1.715 8966	+0.134 0934	+1.709 5212	+0.127 0511
33	+1.700 3702	+0.219 3560	+1.695 9944	+0.207 6356	+1.689 4904	+0.200 4263
34	+1.673 0827	+0.290 4526	+1.668 4777	+0.278 6726	+1.661 8933	+0.271 2560
35	+1.638 4553	+0.358 3016	+1.633 6597	+0.346 4130	+1.627 0482	+0.338 7587
36	+1.596 8743	+0.422 1439	+1.591 9353	+0.410 1049	+1.585 3540	+0.402 1930
37	+1.548 8048	+0.481 2659	+1.543 7778	+0.469 0434	+1.537 2837	+0.460 8658
38	+1.494 7865	+0.535 0089	+1.489 7309	+0.522 5796	+1.483 3802	+0.514 1397
39	+1.435 4258	+0.582 7755	+1.430 4045	+0.570 1266	+1.424 2480	+0.561 4397
40	+1.371 3900	+0.624 0380	+1.366 4651	+0.611 1675	+1.360 5472	+0.602 2588
41	+1.303 3980	+0.658 3431	+1.298 6290	+0.645 2602	+1.292 9865	+0.636 1634
42	+1.232 2125	+0.685 3171	+1.227 6529	+0.672 0417	+1.222 3115	+0.662 7978
43	+1.158 6310	+0.704 6698	+1.154 3266	+0.691 2326	+1.149 3005	+0.681 8865
44	+1.083 4752	+0.716 1979	+1.079 4622	+0.702 6377	+1.074 7540	+0.693 2381
45	+1.007 5811	+0.719 7864	+1.003 8858	+0.706 1489	+0.999 4847	+0.696 7453
46	+0.931 7909	+0.715 4100	+0.928 4277	+0.701 7444	+0.924 3115	+0.692 3855
47	+0.856 9412	+0.703 1324	+0.853 9128	+0.689 4895	+0.850 0486	+0.680 2209
48	+0.783 8553	+0.683 1047	+0.781 1516	+0.669 5340	+0.777 4975	+0.660 3960
49	+0.713 3338	+0.655 5634	+0.710 9317	+0.642 1111	+0.707 4383	+0.633 1359
50	+0.646 1446	+0.620 8269	+0.644 0104	+0.607 5322	+0.640 6223	+0.598 7436
51	+0.583 0165	+0.579 2912	+0.581 1065	+0.566 1862	+0.577 7638	+0.557 5971
52	+0.524 6297	+0.531 4234	+0.522 8929	+0.518 5316	+0.519 5340	+0.510 1441
53	+0.471 6110	+0.477 7588	+0.469 9907	+0.465 0927	+0.466 5545	+0.456 8979
54	+0.424 5246	+0.418 8929	+0.422 9616	+0.406 4538	+0.419 3902	+0.398 4327
55	+0.383 8706	+0.355 4747	+0.382 3050	+0.343 2525	+0.378 5456	+0.335 3763
56	+0.350 0760	+0.288 2002	+0.348 4501	+0.276 1738	+0.344 4581	+0.268 4051
57	+0.323 4952	+0.217 8039	+0.321 7545	+0.205 9419	+0.317 4943	+0.198 2368
58	+0.304 4032	+0.145 0512	+0.302 4996	+0.133 3140	+0.297 9461	+0.125 6232
59	+0.292 9944	+0.070 7305	+0.290 8879	+0.059 0724	+0.286 0278	+0.051 3438

TABLE V.—*Concluded.*

RECTANGULAR G-COÖRDINATES X AND Y OF VENUS, REFERRED, IN EACH SYSTEM, TO AN AXIS OF X PASSING THROUGH THE MEAN SUN.

i	System 9.		System 10.		System 11.	
	X	*Y*	*X*	*Y*	*X*	*Y*
0	+0.275 0523	−0.025 3633	+0.268 5973	−0.019 7819	+0.264 6365	−0.008 4260
1	+0.278 4844	−0.100 7462	+0.271 9768	−0.095 3562	+0.268 1640	−0.084 2045
2	+0.289 7521	−0.175 4000	+0.283 2537	−0.170 2047	+0.279 6432	−0.159 2361
3	+0.308 7484	−0.248 5086	+0.302 3220	−0.243 5018	+0.298 9610	−0.232 6860
4	+0.335 2831	−0.319 2704	+0.328 9878	−0.314 4354	+0.325 9149	−0.303 7353
5	+0.369 0820	−0.386 9062	+0.362 9727	−0.382 2158	+0.360 2162	−0.371 5891
6	+0.409 7903	−0.450 6682	+0.403 9145	−0.446 0864	+0.401 4013	−0.435 4874
7	+0.456 9758	−0.509 8485	+0.451 3723	−0.505 3317	+0.449 2861	−0.494 7126
8	+0.510 1331	−0.563 7854	+0.504 8294	−0.559 2848	+0.503 0724	−0.548 5982
9	+0.568 6886	−0.611 8737	+0.563 7000	−0.607 3382	+0.562 2517	−0.596 5386
10	+0.632 0062	−0.653 5698	+0.627 3358	−0.648 9481	+0.626 1636	−0.637 9955
11	+0.699 3951	−0.688 4000	+0.695 0329	−0.683 6429	+0.694 0946	−0.672 5042
12	+0.770 1160	−0.715 9656	+0.766 0396	−0.711 0281	+0.765 2841	−0.699 6795
13	+0.843 3898	−0.735 9488	+0.839 5657	−0.730 7919	+0.838 9356	−0.719 2203
14	+0.918 4056	−0.748 1159	+0.914 7912	−0.742 7091	+0.914 2243	−0.730 9118
15	+0.994 3301	−0.752 3209	+0.990 8753	−0.746 6441	+0.990 3091	−0.734 6303
16	+1.070 3171	−0.748 5084	+1.066 9676	−0.742 5523	+1.066 3406	−0.730 3419
17	+1.145 5178	−0.736 7126	+1.142 2168	−0.730 4812	+1.141 4717	−0.718 1042
18	+1.219 0910	−0.717 0597	+1.215 7813	−0.710 5689	+1.214 8681	−0.698 0645
19	+1.290 2123	−0.689 7649	+1.286 8399	−0.683 0438	+1.285 7175	−0.670 4585
20	+1.358 0846	−0.655 1320	+1.354 5999	−0.648 2193	+1.353 2374	−0.635 6060
21	+1.421 9480	−0.613 5482	+1.418 3087	−0.606 4921	+1.416 6861	−0.593 9072
22	+1.481 0876	−0.565 4803	+1.477 2619	−0.558 3354	+1.475 3700	−0.545 8365
23	+1.534 8440	−0.511 4688	+1.530 8092	−0.504 2948	+1.528 6519	−0.491 9379
24	+1.582 6192	−0.452 1210	+1.578 3646	−0.444 9805	+1.575 9570	−0.432 8169
25	+1.623 8856	−0.388 1040	+1.619 4111	−0.381 0590	+1.616 7795	−0.369 1332
26	+1.658 1899	−0.320 1363	+1.653 5064	−0.313 2470	+1.650 6865	−0.301 5947
27	+1.685 1598	−0.248 9795	+1.680 2882	−0.242 3001	+1.677 3221	−0.230 9478
28	+1.704 5055	−0.175 4284	+1.699 4769	−0.169 0056	+1.696 4120	−0.157 9687
29	+1.716 0249	−0.100 3034	+1.710 8776	−0.094 1744	+1.707 7640	−0.083 4561
30	+1.719 6036	−0.024 4386	+1.714 3826	−0.018 6290	+1.711 2698	−0.008 2204
31	+1.715 2172	+0.051 3259	+1.709 9702	+0.056 8022	+1.706 9075	+0.066 9225
32	+1.702 9292	+0.126 1551	+1.697 7055	+0.131 2957	+1.694 7377	+0.141 1607
33	+1.682 8904	+0.199 2270	+1.677 7386	+0.204 0425	+1.674 9043	+0.213 6945
34	+1.655 3370	+0.269 7422	+1.650 3021	+0.274 2547	+1.647 6325	+0.283 7444
35	+1.620 5873	+0.336 9322	+1.615 7077	+0.341 1756	+1.613 2252	+0.350 5588
36	+1.579 0366	+0.400 0673	+1.574 3442	+0.404 0865	+1.572 0603	+0.413 4229
37	+1.531 1536	+0.458 4650	+1.526 6706	+0.462 3116	+1.524 5861	+0.471 6640
38	+1.477 4737	+0.511 4975	+1.473 2125	+0.515 2296	+1.471 3167	+0.524 6595
39	+1.418 5926	+0.558 5979	+1.414 5550	+0.562 2757	+1.412 8279	+0.571 8423
40	+1.355 1605	+0.599 2658	+1.351 3371	+0.602 9494	+1.349 7496	+0.612 7065
41	+1.287 8735	+0.633 0725	+1.284 2449	+0.636 8197	+1.282 7595	+0.646 8129
42	+1.217 4671	+0.659 6638	+1.214 0031	+0.663 5280	+1.212 5774	+0.673 7929
43	+1.144 7074	+0.678 7636	+1.141 3690	+0.682 7924	+1.139 9554	+0.693 3538
44	+1.070 3837	+0.690 1770	+1.067 1246	+0.694 4103	+1.065 6731	+0.705 2806
45	+0.995 2985	+0.693 7915	+0.992 0676	+0.698 2589	+0.990 5284	+0.709 4400
46	+0.920 2620	+0.689 5775	+0.917 0042	+0.694 2980	+0.915 3290	+0.705 7790
47	+0.846 0814	+0.677 5884	+0.842 7410	+0.682 5694	+0.840 8858	+0.694 3282
48	+0.773 5541	+0.657 9598	+0.770 0760	+0.663 1965	+0.768 0039	+0.675 2018
49	+0.703 4583	+0.630 9076	+0.699 7907	+0.636 3845	+0.697 4731	+0.648 5938
50	+0.636 5464	+0.596 7255	+0.632 6437	+0.602 4170	+0.630 0613	+0.614 7816
51	+0.573 5356	+0.555 7804	+0.569 3612	+0.561 6533	+0.566 5049	+0.574 1195
52	+0.515 1027	+0.508 5110	+0.510 6299	+0.514 5258	+0.507 5018	+0.527 0362
53	+0.461 8752	+0.455 4212	+0.457 0893	+0.461 5341	+0.453 7038	+0.474 0329
54	+0.414 4269	+0.397 0772	+0.409 3243	+0.403 2415	+0.405 7084	+0.415 6762
55	+0.373 2719	+0.334 1012	+0.367 8611	+0.340 2695	+0.364 0524	+0.352 5927
56	+0.338 8588	+0.267 1652	+0.333 1597	+0.273 2907	+0.329 2053	+0.285 4627
57	+0.311 5656	+0.196 9846	+0.305 6091	+0.203 0240	+0.301 5630	+0.215 0136
58	+0.291 6964	+0.124 3117	+0.285 5228	+0.130 2265	+0.281 4442	+0.142 0126
59	+0.279 4770	+0.049 9280	+0.273 1348	+0.055 6874	+0.269 0848	+0.067 2585

TABLE VI.

G-COÖRDINATE Z OF VENUS.

System *i*	0	1	2	3	4	5
0	+.017 4641	+.034 6635	+.042 3898	+.038 5648	+.024 3031	+.003 5232
1	+.021 4657	+.037 0784	+.042 5597	+.036 4563	+.020 5041	−.000 9384
2	+.025 2276	+.039 0785	+.042 2557	+.033 9453	+.016 4810	−.005 3894
3	+.028 7075	+.040 6417	+.041 4818	+.031 0603	+.012 2785	−.009 7824
4	+.031 8665	+.041 7509	+.040 2471	+.027 8339	+.007 9423	−.014 0692
5	+.034 6689	+.042 3939	+.038 5664	+.024 3022	+.003 5202	−.018 2044
6	+.037 0834	+.042 5642	+.036 4589	+.020 5043	−.000 9400	−.022 1436
7	+.039 0828	+.042 2603	+.033 9486	+.016 4824	−.005 3898	−.025 8445
8	+.040 6450	+.041 4861	+.031 0641	+.012 2809	−.009 7816	−.029 2675
9	+.041 7527	+.040 2507	+.027 8378	+.007 9454	−.014 0672	−.032 3763
10	+.042 3943	+.038 5690	+.024 3057	+.003 5235	−.018 2018	−.035 1374
11	+.042 5630	+.036 4602	+.020 5072	−.000 9368	−.022 1406	−.037 5214
12	+.042 2579	+.033 9485	+.016 4842	−.005 3871	−.025 8416	−.039 5028
13	+.041 4828	+.031 0627	+.012 2814	−.009 7798	−.029 2651	−.041 0601
14	+.040 2473	+.027 8355	+.007 9446	−.014 0665	−.032 3746	−.042 1764
15	+.038 5660	+.024 3029	+.003 5218	−.018 2022	−.035 1366	−.042 8392
16	+.036 4580	+.020 5046	−.000 9389	−.022 1419	−.037 5215	−.043 0410
17	+.033 9476	+.016 4822	−.005 3893	−.025 8434	−.039 5037	−.042 7792
18	+.031 0631	+.012 2804	−.009 7815	−.029 2671	−.041 0620	−.042 0556
19	+.027 8370	+.007 9450	−.014 0675	−.032 3762	−.042 1783	−.040 8775
20	+.024 3054	+.003 5233	−.018 2021	−.035 1377	−.042 8411	−.039 2568
21	+.020 5072	−.000 9367	−.022 1407	−.037 5218	−.043 0424	−.037 2103
22	+.016 4847	−.005 3866	−.025 8415	−.039 5032	−.042 7800	−.034 7593
23	+.012 2824	−.009 7789	−.029 2648	−.041 0604	−.042 0556	−.031 9296
24	+.007 9461	−.014 0654	−.032 3739	−.042 1765	−.040 8769	−.028 7515
25	+.003 5237	−.018 2007	−.035 1357	−.042 8391	−.039 2556	−.025 2590
26	−.000 9370	−.022 1401	−.037 5204	−.043 0406	−.037 2089	−.021 4899
27	−.005 3875	−.025 8415	−.039 5023	−.042 7786	−.034 7578	−.017 4848
28	−.009 7799	−.029 2652	−.041 0600	−.042 0548	−.031 9285	−.013 2876
29	−.014 0663	−.032 3746	−.042 1766	−.040 8766	−.028 7508	−.008 9440
30	−.018 2014	−.035 1364	−.042 8395	−.039 2557	−.025 2588	−.004 5014
31	−.022 1404	−.037 5209	−.043 0411	−.037 2091	−.021 4900	−.000 0091
32	−.025 8416	−.039 5026	−.042 7789	−.034 7581	−.017 4851	+.004 4833
33	−.029 2651	−.041 0601	−.042 0550	−.031 9287	−.013 2878	+.008 9260
34	−.032 3744	−.042 1765	−.040 8765	−.028 7508	−.008 9438	+.013 2696
35	−.035 1363	−.042 8394	−.039 2555	−.025 2585	−.004 5009	+.017 4653
36	−.037 5212	−.043 0411	−.037 2089	−.021 4895	−.000 0084	+.021 4663
37	−.039 5032	−.042 7792	−.034 7581	−.017 4846	+.004 4842	+.025 2276
38	−.041 0611	−.042 0556	−.031 9288	−.013 2874	+.008 9269	+.028 7069
39	−.042 1777	−.040 8775	−.028 7513	−.008 9438	+.013 2702	+.031 8651
40	−.042 8407	−.039 2567	−.025 2594	−.004 5015	+.017 4654	+.034 6668
41	−.043 0424	−.037 2103	−.021 4908	−.000 0095	+.021 4658	+.037 0806
42	−.042 7801	−.034 7594	−.017 4861	+.004 4827	+.025 2265	+.039 0797
43	−.042 0559	−.031 9300	−.013 2889	+.008 9251	+.028 7052	+.040 6420
44	−.040 8771	−.028 7519	−.008 9451	+.013 2684	+.031 8631	+.041 7501
45	−.039 2555	−.025 2594	−.004 5022	+.017 4641	+.034 6648	+.042 3922
46	−.037 2085	−.021 4900	−.000 0095	+.021 4650	+.037 0791	+.042 5618
47	−.034 7572	−.017 4846	+.004 4835	+.025 2265	+.039 0788	+.042 2575
48	−.031 9278	−.013 2869	+.008 9269	+.028 7061	+.040 6418	+.041 4834
49	−.028 7504	−.008 4931	+.013 2707	+.031 8649	+.041 7508	+.040 2487
50	−.025 2588	−.004 5007	+.017 4664	+.034 6672	+.042 3939	+.038 5680
51	−.021 4907	−.000 0088	+.021 4671	+.037 0818	+.042 5641	+.036 4606
52	−.017 4867	+.004 4829	+.025 2277	+.039 0815	+.042 2603	+.033 9503
53	−.013 2904	+.008 9247	+.028 7061	+.040 6439	+.041 4862	+.031 0659
54	−.008 9473	+.013 2672	+.031 8635	+.041 7518	+.040 2511	+.027 8396
55	−.004 5052	+.017 4621	+.034 6645	+.042 3935	+.038 5697	+.024 3076
56	−.000 0130	+.021 4624	+.037 0779	+.042 5624	+.036 4611	+.020 5092
57	+.004 4797	+.025 2232	+.039 0768	+.042 2573	+.033 9495	+.016 4863
58	+.008 9228	+.028 7024	+.040 6390	+.041 4822	+.031 0636	+.012 2834
59	+.013 2669	+.031 8610	+.041 7474	+.040 2465	+.027 8361	+.007 9465

TABLE VI.—*Concluded.*

G-COÖRDINATE Z OF VENUS.

System *i*	6	7	8	9	10	11
0	—.018 2000	—.035 1342	—.042 8369	—.039 2532	—.025 2572	—.004 5010
1	—.022 1401	—.037 5194	—.043 0386	—.037 2062	—.021 4874	—.000 0076
2	—.025 8423	—.039 5018	—.042 7767	—.034 7550	—.017 4818	+.004 4858
3	—.029 2668	—.041 0601	—.042 0532	—.031 9256	—.013 2840	+.008 9294
4	—.032 3768	—.042 1773	—.040 8754	—.028 7481	—.008 9400	+.013 2734
5	—.035 1389	—.042 8410	—.039 2552	—.025 2564	—.004 4976	+.017 4692
6	—.037 5236	—.043 0430	—.037 2094	—.021 4882	—.000 0058	+.021 4697
7	—.039 5053	—.042 7812	—.034 7591	—.017 4841	+.004 4860	+.025 2302
8	—.041 0626	—.042 0571	—.031 9302	—.013 2877	+.008 9278	+.028 7085
9	—.042 1786	—.040 8785	—.028 7525	—.008 9446	+.013 2704	+.031 8658
10	—.042 8409	—.039 2571	—.025 2602	—.004 5023	+.017 4650	+.034 6667
11	—.043 0419	—.037 2100	—.021 4909	—.000 0100	+.021 4652	+.037 0801
12	—.042 7792	—.034 7584	—.017 4855	+.004 4828	+.025 2260	+.039 0788
13	—.042 0547	—.031 9285	—.013 2877	+.008 9260	+.028 7051	+.040 6410
14	—.040 8759	—.028 7504	—.008 9434	+.013 2698	+.031 8637	+.041 7493
15	—.039 2548	—.025 2577	—.004 5004	+.017 4658	+.034 6659	+.042 3920
16	—.037 2083	—.021 4886	—.000 0078	+.021 4667	+.037 0804	+.042 5621
17	—.034 7576	—.017 4838	+.004 4849	+.025 2279	+.039 0801	+.042 2582
18	—.031 9285	—.013 2868	+.008 9276	+.028 7070	+.040 6429	+.041 4843
19	—.028 7511	—.008 9433	+.013 2707	+.031 8652	+.041 7515	+.040 2494
20	—.025 2591	—.004 5010	+.017 4659	+.034 6669	+.042 3938	+.038 5683
21	—.021 4905	—.000 0089	+.021 4663	+0.37 0806	+.042 5634	+.036 4601
22	—.017 4858	+.004 4834	+.025 2269	+.039 0798	+.042 2588	+.033 9492
23	—.013 2884	+.008 9260	+.028 7058	+.040 6420	+.041 4841	+.031 0639
24	—.008 9445	+.013 2693	+.031 8639	+.041 7503	+.040 2485	+.027 8371
25	—.004 5016	+.017 4648	+.034 6657	+.042 3926	+.038 5671	+.024 3046
26	—.000 0090	+.021 4657	+.037 0798	+.042 5622	+.036 4589	+.020 5061
27	+.004 4836	+.025 2267	+.039 0792	+.042 2578	+.033 9480	+.016 4835
28	+.008 9263	+.028 7059	+.040 6418	+.041 4834	+.031 0630	+.012 2812
29	+.013 2697	+.031 8640	+.041 7503	+.040 2483	+.027 8365	+.007 9453
30	+.017 4651	+.034 6658	+.042 3926	+.038 5671	+.024 3044	+.003 5232
31	+.021 4656	+.037 0798	+.042 5622	+.036 4589	+.020 5061	—.000 9372
32	+.025 2265	+.039 0790	+.042 2577	+.033 9481	+.016 4835	—.005 3874
33	+.028 7055	+.040 6414	+.041 4832	+.031 0629	+.012 2813	—.009 7798
34	+.031 8637	+.041 7499	+.040 2479	+.027 8362	+.007 9452	—.014 0663
35	+.034 6657	+.042 3924	+.038 5666	+.024 3039	+.003 5227	—.018 2016
36	+.037 0803	+.042 5624	+.036 4587	+.020 5053	—.000 9380	—.022 1410
37	+.039 0800	+.042 2582	+.033 9481	+.016 4829	—.005 3885	—.025 8425
38	+.040 6429	+.041 4843	+.031 0634	+.012 2810	—.009 7809	—.029 2663
39	+.041 7516	+.040 2496	+.027 8373	+.007 9454	—.014 0673	—.032 3757
40	+.042 3941	+.038 5686	+.024 3059	+.003 5236	—.018 2018	—.035 1375
41	+.042 5637	+.036 4607	+.020 5076	—.000 9364	—.022 1405	—.037 5218
42	+.042 2590	+.033 9500	+.016 4854	—.005 3861	—.025 8411	—.039 5031
43	+.041 4843	+.031 0647	+.012 2833	—.009 7782	—.029 2643	—.041 0603
44	+.040 2488	+.027 8377	+.007 9472	—.014 0645	—.032 3732	—.042 1761
45	+.038 5672	+.024 3051	+.003 5245	—.018 1997	—.035 1349	—.042 8386
46	+.036 4590	+.020 5062	—.000 9367	—.022 1394	—.037 5197	—.043 0401
47	+.033 9480	+.016 4835	—.005 3875	—.025 8414	—.039 5020	—.042 7782
48	+.031 0631	+.012 2812	—.009 7805	—.029 2657	—.041 0604	—.042 0548
49	+.027 8367	+.007 9452	—.014 0671	—.032 3758	—.042 1777	—.040 8770
50	+.024 3049	+.003 5231	—.018 2023	—.035 1381	—.042 8413	—.039 2569
51	+.020 5071	—.000 9371	—.022 1413	—.037 5229	—.043 0437	—.037 2112
52	+.016 4852	—.005 3869	—.025 8422	—.039 5045	—0.42 7820	—.034 7610
53	+.012 2837	—.009 7786	—.029 2654	—.041 0619	—.042 0581	—.031 9323
54	+.007 9482	—.014 0642	—.032 3741	—.042 1778	—.040 8794	—.028 7548
55	+.003 5264	—.018 1987	—.035 1353	—.042 8401	—.039 2580	—.025 2624
56	—.000 9340	—.022 1376	—.037 5193	—.043 0411	—.037 2108	—.021 4931
57	—.005 3844	—.025 8388	—.039 5006	—.042 7784	—.034 7591	—.017 4875
58	—.009 7771	—.029 2624	—.041 0578	—.042 0538	—.031 9290	—.013 2894
59	—.014 0642	—.032 3720	—.042 1740	—.040 8748	—.028 7503	—.008 9447

TABLE VII.

Values of A, B, C, and D for the Action of Venus on the Moon.

	System 0.					System 1.			
i	*A*	*B*	*C*	*D*	*i*	*A*	*B*	*C*	*D*
0	+35.541 47	−17.876 47	−17.664 99	+ 1.074 27	0	+32.593 73	−16.607 73	−15.985 99	+ 3.187 45
1	+27.684 74	−12.483 21	−15.201 55	−11.369 32	1	+27.372 10	−13.088 33	−14.283 78	− 9.177 80
2	+14.109 43	− 3.818 23	−10.291 18	−12.812 04	2	+14.893 21	− 4.857 17	−10.036 02	−11.992 56
3	+ 6.072 22	+ 0.262 35	− 6.334 56	− 9.167 52	3	+ 6.704 15	− 0.374 85	− 6.329 30	− 9.064 78
4	+ 2.639 36	+ 1.251 33	− 3.890 71	− 5.857 72	4	+ 3.003 34	+ 0.935 15	− 3.938 50	− 5.930 21
5	+ 1.270 90	+ 1.203 98	− 2.474 87	− 3.746 92	5	+ 1.474 29	+ 1.046 97	− 2.521 25	− 3.830 19
6	+ 0.707 97	+ 0.937 76	− 1.645 74	− 2.486 04	6	+ 0.827 46	+ 0.853 85	− 1.681 30	− 2.549 54
7	+ 0.459 96	+ 0.682 85	− 1.142 81	− 1.722 74	7	+ 0.535 00	+ 0.633 64	− 1.168 65	− 1.767 04
8	+ 0.341 13	+ 0.484 05	− 0.825 17	− 1.243 31	8	+ 0.391 31	+ 0.452 53	− 0.843 83	− 1.273 60
9	+ 0.278 66	+ 0.337 95	− 0.616 61	− 0.929 36	9	+ 0.314 04	+ 0.316 17	− 0.630 22	− 0.950 06
10	+ 0.242 48	+ 0.232 28	− 0.474 76	− 0.715 40	10	+ 0.258 52	+ 0.216 33	− 0.484 84	− 0.729 59
11	+ 0.219 41	+ 0.155 84	− 0.375 25	− 0.564 21	11	+ 0.239 23	+ 0.143 61	− 0.382 83	− 0.573 95
12	+ 0.203 37	+ 0.100 16	− 0.303 52	− 0.453 90	12	+ 0.218 85	+ 0.090 46	− 0.309 32	− 0.460 55
13	+ 0.191 38	+ 0.059 22	− 0.250 58	− 0.371 12	13	+ 0.203 73	+ 0.051 35	− 0.255 07	− 0.375 58
14	+ 0.181 90	+ 0.028 81	− 0.210 71	− 0.307 44	14	+ 0.191 91	+ 0.022 31	− 0.214 22	− 0.310 32
15	+ 0.174 12	+ 0.006 01	− 0.180 13	− 0.257 36	15	+ 0.182 33	+ 0.000 58	− 0.182 90	− 0.259 10
16	+ 0.167 56	− 0.011 23	− 0.156 32	− 0.217 19	16	+ 0.174 35	− 0.015 82	− 0.158 53	− 0.218 09
17	+ 0.161 93	− 0.024 38	− 0.137 55	− 0.184 39	17	+ 0.167 57	− 0.028 27	− 0.139 30	− 0.184 66
18	+ 0.157 05	− 0.034 47	− 0.122 58	− 0.157 18	18	+ 0.161 75	− 0.037 78	− 0.123 98	− 0.156 97
19	+ 0.152 80	− 0.042 26	− 0.110 54	− 0.134 23	19	+ 0.156 72	− 0.045 08	− 0.111 65	− 0.133 68
20	+ 0.149 09	− 0.048 31	− 0.100 78	− 0.114 61	20	+ 0.152 35	− 0.050 68	− 0.101 66	− 0.113 79
21	+ 0.145 86	− 0.053 01	− 0.092 86	− 0.097 60	21	+ 0.148 54	− 0.055 01	− 0.093 54	− 0.096 58
22	+ 0.143 07	− 0.056 66	− 0.086 40	− 0.082 65	22	+ 0.145 24	− 0.058 33	− 0.086 92	− 0.081 48
23	+ 0.140 67	− 0.059 51	− 0.081 17	− 0.069 33	23	+ 0.142 39	− 0.060 86	− 0.081 53	− 0.068 04
24	+ 0.138 64	− 0.061 71	− 0.076 93	− 0.057 30	24	+ 0.139 95	− 0.062 78	− 0.077 17	− 0.055 94
25	+ 0.136 93	− 0.063 38	− 0.073 56	− 0.046 29	25	+ 0.137 89	− 0.064 20	− 0.073 68	− 0.044 87
26	+ 0.135 58	− 0.064 64	− 0.070 94	− 0.036 06	26	+ 0.136 16	− 0.065 22	− 0.070 96	− 0.034 61
27	+ 0.134 53	− 0.065 55	− 0.068 98	− 0.026 43	27	+ 0.134 78	− 0.065 89	− 0.068 90	− 0.024 97
28	+ 0.133 77	− 0.066 15	− 0.067 61	− 0.017 22	28	+ 0.133 71	− 0.066 26	− 0.067 44	− 0.015 78
29	+ 0.133 30	− 0.066 49	− 0.066 82	− 0.008 27	29	+ 0.132 94	− 0.066 38	− 0.066 55	− 0.006 88
30	+ 0.133 12	− 0.066 57	− 0.066 55	+ 0.000 51	30	+ 0.132 45	− 0.066 26	− 0.066 20	+ 0.001 87
31	+ 0.133 22	− 0.066 41	− 0.066 81	+ 0.009 30	31	+ 0.132 25	− 0.065 89	− 0.066 36	+ 0.010 60
32	+ 0.133 61	− 0.066 00	− 0.067 60	+ 0.018 23	32	+ 0.132 34	− 0.065 28	− 0.067 06	+ 0.019 44
33	+ 0.134 28	− 0.065 32	− 0.068 96	+ 0.027 44	33	+ 0.132 70	− 0.064 38	− 0.068 31	+ 0.028 54
34	+ 0.135 25	− 0.064 33	− 0.070 91	+ 0.037 06	34	+ 0.133 34	− 0.063 19	− 0.070 16	+ 0.038 03
35	+ 0.136 51	− 0.062 99	− 0.073 53	+ 0.047 28	35	+ 0.134 27	− 0.061 63	− 0.072 64	+ 0.048 08
36	+ 0.138 10	− 0.061 21	− 0.076 88	+ 0.058 27	36	+ 0.135 49	− 0.059 63	− 0.075 86	+ 0.058 87
37	+ 0.140 01	− 0.058 92	− 0.081 09	+ 0.070 27	37	+ 0.137 02	− 0.057 10	− 0.079 91	+ 0.070 63
38	+ 0.142 27	− 0.055 97	− 0.086 30	+ 0.083 55	38	+ 0.138 86	− 0.053 92	− 0.084 95	+ 0.083 61
39	+ 0.144 90	− 0.052 19	− 0.092 72	+ 0.098 44	39	+ 0.141 03	− 0.049 89	− 0.091 15	+ 0.098 14
40	+ 0.147 95	− 0.047 35	− 0.100 59	+ 0.115 35	40	+ 0.143 56	− 0.044 79	− 0.098 76	+ 0.114 61
41	+ 0.151 43	− 0.041 17	− 0.110 28	+ 0.134 84	41	+ 0.146 46	− 0.038 34	− 0.108 13	+ 0.133 55
42	+ 0.155 43	− 0.033 21	− 0.122 23	+ 0.157 60	42	+ 0.149 78	− 0.030 10	− 0.119 68	+ 0.155 62
43	+ 0.160 00	− 0.022 94	− 0.137 06	+ 0.184 57	43	+ 0.153 57	− 0.019 53	− 0.134 03	+ 0.181 71
44	+ 0.165 26	− 0.009 60	− 0.155 66	+ 0.217 00	44	+ 0.157 89	− 0.005 90	− 0.151 99	+ 0.213 01
45	+ 0.171 35	+ 0.007 85	− 0.179 20	+ 0.256 65	45	+ 0.162 86	+ 0.011 84	− 0.174 71	+ 0.251 18
46	+ 0.178 54	+ 0.030 87	− 0.209 41	+ 0.305 97	46	+ 0.168 67	+ 0.035 14	− 0.203 81	+ 0.298 54
47	+ 0.187 24	+ 0.061 51	− 0.248 75	+ 0.368 55	47	+ 0.175 61	+ 0.066 01	− 0.241 63	+ 0.358 46
48	+ 0.198 20	+ 0.102 69	− 0.300 91	+ 0.449 99	48	+ 0.184 28	+ 0.107 35	− 0.291 64	+ 0.435 90
49	+ 0.212 80	+ 0.158 66	− 0.371 45	+ 0.557 51	49	+ 0.195 77	+ 0.163 31	− 0.359 07	+ 0.538 46
50	+ 0.233 70	+ 0.235 44	− 0.469 14	+ 0.704 84	50	+ 0.212 25	+ 0.239 85	− 0.452 11	+ 0.678 05
51	+ 0.266 43	+ 0.341 67	− 0.608 10	+ 0.912 63	51	+ 0.238 32	+ 0.345 49	− 0.583 82	+ 0.874 09
52	+ 0.322 91	+ 0.489 03	− 0.811 95	+ 1.216 44	52	+ 0.284 04	+ 0.491 88	− 0.775 91	+ 1.159 30
53	+ 0.430 39	+ 0.691 21	− 1.121 59	+ 1.678 67	53	+ 0.372 52	+ 0.693 04	− 1.065 56	+ 1.590 84
54	+ 0.654 64	+ 0.955 89	− 1.610 52	+ 2.412 20	54	+ 0.559 82	+ 0.958 81	− 1.518 64	+ 2.271 42
55	+ 1.163 22	+ 1.251 17	− 2.414 40	+ 3.621 88	55	+ 0.988 62	+ 1.266 17	− 2.254 78	+ 3.386 75
56	+ 2.399 22	+ 1.385 11	− 3.784 32	+ 5.653 79	56	+ 2.034 82	+ 1.456 00	− 3.490 84	+ 5.251 63
57	+ 5.511 93	+ 0.638 08	− 6.150 01	+ 8.900 00	57	+ 4.672 67	+ 0.916 43	− 5.589 09	+ 8.243 78
58	+12.932 86	− 2.923 59	−10.009 26	+12.777 21	58	+11.007 05	− 2.044 99	− 8.962 05	+11.988 39
59	+26.185 67	−11.259 17	−14.926 49	+12.461 23	59	+22.761 17	− 9.475 57	−13.285 60	+12.554 18

TABLE VII.—*Continued.*

VALUES OF A, B, C, AND D FOR THE ACTION OF VENUS ON THE MOON.

System 2.					System 3.				
i	*A*	*B*	*C*	*D*	*i*	*A*	*B*	*C*	*D*
0	+29.624 12	−15.178 31	−14.445 82	+ 3.983 50	0	+28.095 38	−14.290 79	−13.804 59	+ 3.723 68
1	+26.178 17	−12.858 22	−13.319 98	− 7.553 34	1	+25.108 12	−12.240 65	−12.867 46	− 6.979 90
2	+15.058 97	− 5.374 64	− 9.684 31	−11.068 21	2	+14.804 22	− 5.334 68	− 9.469 55	−10.470 31
3	+ 7.082 50	− 0.827 44	− 6.255 07	− 8.775 53	3	+ 7.142 71	− 0.971 49	− 6.171 21	− 8.464 11
4	+ 3.272 10	+ 0.673 17	− 3.945 29	− 5.880 03	4	+ 3.368 04	+ 0.545 38	− 3.913 41	− 5.747 32
5	+ 1.637 83	+ 0.905 32	− 2.543 15	− 3.842 60	5	+ 1.709 07	+ 0.821 00	− 2.530 07	− 3.786 67
6	+ 0.926 81	+ 0.774 63	− 1.701 45	− 2.571 56	6	+ 0.973 32	+ 0.721 94	− 1.695 25	− 2.546 14
7	+ 0.597 76	+ 0.586 40	− 1.184 16	− 1.785 90	7	+ 0.627 53	+ 0.553 13	− 1.180 66	− 1.772 84
8	+ 0.432 83	+ 0.422 37	− 0.855 20	− 1.287 47	8	+ 0.452 10	+ 0.400 75	− 0.852 85	− 1.279 74
9	+ 0.342 73	+ 0.295 72	− 0.638 46	− 0.959 70	9	+ 0.355 45	+ 0.281 22	− 0.636 65	− 0.954 45
10	+ 0.289 09	+ 0.201 73	− 0.490 82	− 0.736 06	10	+ 0.297 60	+ 0.191 71	− 0.489 33	− 0.732 10
11	+ 0.254 43	+ 0.132 76	− 0.387 20	− 0.578 14	11	+ 0.260 20	+ 0.125 69	− 0.385 89	− 0.574 91
12	+ 0.230 36	+ 0.082 14	− 0.312 52	− 0.463 12	12	+ 0.234 28	+ 0.077 06	− 0.311 33	− 0.450 36
13	+ 0.212 60	+ 0.044 82	− 0.257 41	− 0.377 00	13	+ 0.215 22	+ 0.041 10	− 0.256 34	− 0.374 58
14	+ 0.198 84	+ 0.017 09	− 0.215 94	− 0.310 93	14	+ 0.200 55	+ 0.014 37	− 0.214 94	− 0.308 76
15	+ 0.187 78	− 0.003 64	− 0.184 16	− 0.259 12	15	+ 0.188 83	− 0.005 63	− 0.183 21	− 0.257 16
16	+ 0.178 66	− 0.019 25	− 0.159 42	− 0.217 69	16	+ 0.179 21	− 0.020 68	− 0.158 52	− 0.215 91
17	+ 0.170 99	− 0.031 07	− 0.139 91	− 0.183 97	17	+ 0.171 14	− 0.032 08	− 0.139 06	− 0.182 35
18	+ 0.164 44	− 0.040 06	− 0.124 37	− 0.156 06	18	+ 0.164 29	− 0.040 74	− 0.123 55	− 0.154 59
19	+ 0.158 80	− 0.046 93	− 0.111 87	− 0.132 62	19	+ 0.158 41	− 0.047 33	− 0.111 08	− 0.131 28
20	+ 0.153 92	− 0.052 17	− 0.101 74	− 0.112 64	20	+ 0.153 33	− 0.052 36	− 0.100 98	− 0.111 43
21	+ 0.149 68	− 0.056 17	− 0.093 50	− 0.095 37	21	+ 0.148 94	− 0.056 18	− 0.092 76	− 0.094 27
22	+ 0.146 00	− 0.059 21	− 0.086 79	− 0.080 24	22	+ 0.145 14	− 0.059 07	− 0.086 06	− 0.079 25
23	+ 0.142 82	− 0.061 50	− 0.081 32	− 0.066 80	23	+ 0.141 84	− 0.061 24	− 0.080 60	− 0.065 93
24	+ 0.140 09	− 0.063 20	− 0.076 89	− 0.054 70	24	+ 0.139 01	− 0.062 82	− 0.076 17	− 0.053 94
25	+ 0.137 75	− 0.064 41	− 0.073 34	− 0.043 66	25	+ 0.136 59	− 0.063 95	− 0.072 63	− 0.043 00
26	+ 0.135 79	− 0.065 24	− 0.070 55	− 0.033 45	26	+ 0.134 54	− 0.064 70	− 0.069 85	− 0.032 89
27	+ 0.134 17	− 0.065 74	− 0.068 44	− 0.023 86	27	+ 0.132 86	− 0.065 13	− 0.067 73	− 0.023 41
28	+ 0.132 88	− 0.065 95	− 0.066 92	− 0.014 73	28	+ 0.131 50	− 0.065 29	− 0.066 22	− 0.014 39
29	+ 0.131 89	− 0.065 92	− 0.065 98	− 0.005 91	29	+ 0.130 46	− 0.065 21	− 0.065 26	− 0.005 68
30	+ 0.131 21	− 0.065 64	− 0.065 56	+ 0.002 75	30	+ 0.129 72	− 0.064 89	− 0.064 83	− 0.002 87
31	+ 0.130 81	− 0.065 13	− 0.065 67	+ 0.011 37	31	+ 0.129 27	− 0.064 34	− 0.064 92	+ 0.011 38
32	+ 0.130 68	− 0.064 38	− 0.066 31	+ 0.020 09	32	+ 0.129 10	− 0.063 57	− 0.065 54	+ 0.019 98
33	+ 0.130 84	− 0.063 36	− 0.067 49	+ 0.029 05	33	+ 0.129 21	− 0.062 53	− 0.066 69	+ 0.028 80
34	+ 0.131 28	− 0.062 03	− 0.069 25	+ 0.038 38	34	+ 0.129 60	− 0.061 19	− 0.068 42	+ 0.038 00
35	+ 0.131 99	− 0.060 35	− 0.071 65	+ 0.048 25	35	+ 0.130 28	− 0.059 50	− 0.070 77	+ 0.047 71
36	+ 0.132 99	− 0.058 23	− 0.074 76	+ 0.058 83	36	+ 0.131 23	− 0.057 40	− 0.073 84	+ 0.058 14
37	+ 0.134 28	− 0.055 58	− 0.078 69	+ 0.070 35	37	+ 0.132 48	− 0.054 78	− 0.077 71	+ 0.069 47
38	+ 0.135 86	− 0.052 28	− 0.083 59	+ 0.083 04	38	+ 0.134 04	− 0.051 51	− 0.082 52	+ 0.081 97
39	+ 0.137 75	− 0.048 15	− 0.089 62	+ 0.097 23	39	+ 0.135 90	− 0.047 44	− 0.088 46	+ 0.095 94
40	+ 0.139 98	− 0.042 95	− 0.097 02	+ 0.113 30	40	+ 0.138 10	− 0.042 35	− 0.095 76	+ 0.111 77
41	+ 0.142 55	− 0.036 42	− 0.106 13	+ 0.131 75	41	+ 0.140 66	− 0.035 93	− 0.104 73	+ 0.129 93
42	+ 0.145 50	− 0.028 12	− 0.117 37	+ 0.153 23	42	+ 0.143 60	− 0.027 81	− 0.115 80	+ 0.151 08
43	+ 0.148 85	− 0.017 55	− 0.131 31	+ 0.178 57	43	+ 0.146 98	− 0.017 46	− 0.129 53	+ 0.176 04
44	+ 0.152 69	− 0.003 95	− 0.148 75	+ 0.208 95	44	+ 0.150 85	− 0.004 19	− 0.146 68	+ 0.205 95
45	+ 0.157 09	+ 0.013 68	− 0.170 77	+ 0.245 92	45	+ 0.155 33	+ 0.013 00	− 0.168 33	+ 0.242 36
46	+ 0.162 22	+ 0.036 71	− 0.198 94	+ 0.291 73	46	+ 0.160 59	+ 0.035 42	− 0.196 01	+ 0.287 47
47	+ 0.168 34	+ 0.067 12	− 0.235 47	+ 0.349 57	47	+ 0.166 02	+ 0.064 96	− 0.231 89	+ 0.344 43
48	+ 0.175 98	+ 0.107 68	− 0.283 65	+ 0.424 17	48	+ 0.174 89	+ 0.104 27	− 0.279 15	+ 0.417 85
49	+ 0.186 10	+ 0.162 34	− 0.348 43	+ 0.522 71	49	+ 0.185 54	+ 0.157 09	− 0.342 64	+ 0.514 83
50	+ 0.200 72	+ 0.236 76	− 0.437 47	+ 0.656 44	50	+ 0.201 00	+ 0.228 74	− 0.429 76	+ 0.646 33
51	+ 0.224 03	+ 0.338 93	− 0.562 96	+ 0.843 60	51	+ 0.225 68	+ 0.326 64	− 0.552 32	+ 0.830 17
52	+ 0.265 29	+ 0.479 66	− 0.744 95	+ 1.114 72	52	+ 0.269 11	+ 0.460 54	− 0.729 66	+ 1.096 02
53	+ 0.345 71	+ 0.671 72	− 1.017 42	+ 1.522 77	53	+ 0.353 01	+ 0.641 40	− 0.994 42	+ 1.495 08
54	+ 0.516 42	+ 0.923 41	− 1.439 82	+ 2.161 97	54	+ 0.529 02	+ 0.874 36	− 1.403 38	+ 2.117 62
55	+ 0.906 36	+ 1.211 94	− 2.118 29	+ 3.200 54	55	+ 0.025 56	+ 1.131 69	− 2.057 25	+ 3.122 67
56	+ 1.850 77	+ 1.390 39	− 3.241 16	+ 4.919 24	56	+ 1.870 94	+ 1.262 21	− 3.133 16	+ 4.769 08
57	+ 4.204 51	+ 0.910 34	− 5.114 84	+ 7.648 85	57	+ 4.183 37	+ 0.732 85	− 4.916 23	+ 7.341 83
58	+ 9.792 93	− 1.714 51	− 8.078 43	+11.072 42	58	+ 9.552 47	− 1.834 00	− 7.718 48	+10.479 49
59	+20.207 75	− 8.331 97	−11.875 75	+11.864 03	59	+19.331 99	− 8.023 85	−11.308 13	+11.063 99

TABLE VII.—*Continued.*

VALUES OF A, B, C, AND D FOR THE ACTION OF VENUS ON THE MOON.

System 4.					System 5.				
i	*A*	*B*	*C*	*D*	*i*	*A*	*B*	*C*	*D*
0	+27.753 75	−13.940 94	−13.812 81	+ 2.712 93	0	+27.639 90	−13.806 76	−13.833 14	+ 1.162 19
1	+24.108 07	−11.431 47	−12.676 62	− 7.271 93	1	+22.709 15	−10.437 90	−12.271 23	− 8.008 99
2	+14.090 02	− 4.811 77	− 9.228 26	−10.185 94	2	+12.869 67	− 4.118 27	− 8.751 42	−10.017 52
3	+ 6.826 96	− 0.838 85	− 5.988 13	− 8.143 97	3	+ 6.173 36	− 0.530 43	− 5.642 95	− 7.784 12
4	+ 3.245 71	+ 0.550 49	− 3.796 20	− 5.536 29	4	+ 2.934 11	+ 0.648 51	− 3.582 61	− 5.261 65
5	+ 1.659 31	+ 0.798 41	− 2.457 70	− 3.661 70	5	+ 1.504 82	+ 0.824 85	− 2.329 66	− 3.486 48
6	+ 0.949 72	+ 0.700 16	− 1.649 89	− 2.471 69	6	+ 0.865 47	+ 0.706 39	− 1.571 85	− 2.362 90
7	+ 0.613 90	+ 0.537 34	− 1.151 26	− 1.726 69	7	+ 0.562 89	+ 0.539 22	− 1.102 12	− 1.658 08
8	+ 0.442 70	+ 0.390 37	− 0.833 07	− 1.249 77	8	+ 0.408 83	+ 0.392 14	− 0.800 98	− 1.205 16
9	+ 0.348 10	+ 0.274 76	− 0.622 86	− 0.934 13	9	+ 0.323 90	+ 0.277 25	− 0.601 14	− 0.904 24
10	+ 0.291 44	+ 0.187 92	− 0.479 37	− 0.717 80	10	+ 0.273 16	+ 0.191 03	− 0.464 18	− 0.697 23
11	+ 0.254 83	+ 0.123 66	− 0.378 48	− 0.564 54	11	+ 0.240 42	+ 0.127 12	− 0.367 55	− 0.550 08
12	+ 0.229 48	+ 0.076 19	− 0.305 68	− 0.452 65	12	+ 0.217 78	+ 0.079 80	− 0.297 59	− 0.442 33
13	+ 0.210 89	+ 0.041 01	− 0.251 91	− 0.368 74	13	+ 0.201 16	+ 0.044 61	− 0.245 78	− 0.361 30
14	+ 0.196 60	+ 0.014 79	− 0.211 39	− 0.304 29	14	+ 0.188 36	+ 0.018 28	− 0.206 64	− 0.298 91
15	+ 0.185 20	− 0.004 89	− 0.180 31	− 0.253 70	15	+ 0.178 13	− 0.001 56	− 0.176 56	− 0.249 82
16	+ 0.175 85	− 0.019 74	− 0.156 12	− 0.213 22	16	+ 0.169 73	− 0.016 61	− 0.153 10	− 0.210 47
17	+ 0.168 03	− 0.031 01	− 0.137 03	− 0.180 25	17	+ 0.162 67	− 0.028 11	− 0.134 57	− 0.178 36
18	+ 0.161 41	− 0.039 59	− 0.121 80	− 0.152 97	18	+ 0.156 68	− 0.036 92	− 0.119 77	− 0.151 73
19	+ 0.155 71	− 0.046 15	− 0.109 56	− 0.130 05	19	+ 0.151 53	− 0.043 69	− 0.107 85	− 0.129 33
20	+ 0.150 82	− 0.051 17	− 0.099 64	− 0.110 51	20	+ 0.147 10	− 0.048 92	− 0.098 19	− 0.110 20
21	+ 0.146 57	− 0.055 01	− 0.091 56	− 0.093 62	21	+ 0.143 28	− 0.052 95	− 0.090 31	− 0.093 64
22	+ 0.142 90	− 0.057 92	− 0.084 98	− 0.078 83	22	+ 0.139 97	− 0.056 07	− 0.083 90	− 0.079 12
23	+ 0.139 75	− 0.060 12	− 0.079 62	− 0.065 70	23	+ 0.137 13	− 0.058 47	− 0.078 68	− 0.066 21
24	+ 0.137 03	− 0.061 75	− 0.075 28	− 0.053 88	24	+ 0.134 73	− 0.060 28	− 0.074 44	− 0.054 57
25	+ 0.134 73	− 0.062 93	− 0.071 80	− 0.043 10	25	+ 0.132 70	− 0.061 64	− 0.071 06	− 0.043 94
26	+ 0.132 80	− 0.063 74	− 0.069 07	− 0.033 13	26	+ 0.131 03	− 0.062 62	− 0.068 42	− 0.034 09
27	+ 0.131 22	− 0.064 23	− 0.067 00	− 0.023 77	27	+ 0.129 70	− 0.063 28	− 0.066 41	− 0.024 84
28	+ 0.129 97	− 0.064 45	− 0.065 51	− 0.014 86	28	+ 0.128 69	− 0.063 69	− 0.065 00	− 0.016 02
29	+ 0.129 04	− 0.064 44	− 0.064 59	− 0.006 25	29	+ 0.127 98	− 0.063 85	− 0.064 13	− 0.007 48
30	+ 0.128 40	− 0.064 21	− 0.064 19	+ 0.002 19	30	+ 0.127 58	− 0.063 79	− 0.063 79	+ 0.000 92
31	+ 0.128 06	− 0.063 75	− 0.064 30	+ 0.010 61	31	+ 0.127 47	− 0.063 51	− 0.063 96	+ 0.009 30
32	+ 0.127 99	− 0.063 06	− 0. 64 94	+ 0.019 12	32	+ 0.127 64	− 0.063 00	− 0.064 65	+ 0.017 79
33	+ 0.128 22	− 0.062 12	− 0.066 11	+ 0.027 87	33	+ 0.128 12	− 0.062 25	− 0.065 87	+ 0.026 52
34	+ 0.128 75	− 0.060 89	− 0.067 84	+ 0.036 98	34	+ 0.128 90	− 0.061 22	− 0.067 68	+ 0.035 65
35	+ 0.129 55	− 0.059 33	− 0.070 22	+ 0.046 63	35	+ 0.129 98	− 0.059 87	− 0.070 11	+ 0.045 33
36	+ 0.130 66	− 0.057 36	− 0.073 29	+ 0.056 99	36	+ 0.131 39	− 0.058 12	− 0.073 26	+ 0.055 75
37	+ 0.132 07	− 0.054 90	− 0.077 18	+ 0.068 27	37	+ 0.133 13	− 0.055 89	− 0.077 23	+ 0.067 11
38	+ 0.133 81	− 0.051 82	− 0.082 00	+ 0.080 73	38	+ 0.135 23	− 0.053 05	− 0.082 17	+ 0.079 68
39	+ 0.135 90	− 0.047 94	− 0.087 95	+ 0.094 66	39	+ 0.137 72	− 0.049 45	− 0.088 25	+ 0.093 78
40	+ 0.138 34	− 0.043 08	− 0.095 27	+ 0.110 47	40	+ 0.140 62	− 0.044 89	− 0.095 73	+ 0.109 81
41	+ 0.141 20	− 0.036 93	− 0.104 26	+ 0.128 64	41	+ 0.144 00	− 0.039 07	− 0.104 93	+ 0.128 28
42	+ 0.144 49	− 0.029 13	− 0.115 36	+ 0.149 82	42	+ 0.147 93	− 0.031 63	− 0.116 28	+ 0.149 86
43	+ 0.148 29	− 0.019 17	− 0.129 14	+ 0.174 86	43	+ 0.152 47	− 0.022 08	− 0.130 40	+ 0.175 44
44	+ 0.152 71	− 0.006 34	− 0.146 36	+ 0.204 92	44	+ 0.157 79	− 0.009 72	− 0.148 08	+ 0.206 20
45	+ 0.157 85	+ 0.010 28	− 0.168 13	+ 0.241 57	45	+ 0.164 07	+ 0.006 38	− 0.170 44	+ 0.243 82
46	+ 0.163 99	+ 0.032 00	− 0.195 97	+ 0.287 04	46	+ 0.171 62	+ 0.027 48	− 0.199 11	+ 0.290 63
47	+ 0.171 48	+ 0.060 63	− 0.232 09	+ 0.344 55	47	+ 0.180 98	+ 0.055 39	− 0.236 37	+ 0.349 98
48	+ 0.181 02	+ 0.098 73	− 0.279 74	+ 0.418 83	48	+ 0.193 04	+ 0.092 61	− 0.285 64	+ 0.426 86
49	+ 0.193 91	+ 0.149 90	− 0.343 80	+ 0.517 08	49	+ 0.209 45	+ 0.142 64	− 0.352 07	+ 0.528 85
50	+ 0.212 65	+ 0.219 19	− 0.431 84	+ 0.650 55	50	+ 0.233 30	+ 0.210 35	− 0.443 66	+ 0.667 81
51	+ 0.242 36	+ 0.313 54	− 0.555 90	+ 0.837 40	51	+ 0.270 84	+ 0.302 33	− 0.573 16	+ 0.862 91
52	+ 0.293 93	+ 0.441 78	− 0.735 71	+ 1.107 89	52	+ 0.335 07	+ 0.426 52	− 0.761 61	+ 1.146 11
53	+ 0.391 71	+ 0.612 99	− 1.004 72	+ 1.514 06	53	+ 0.454 69	+ 0.590 01	− 1.044 70	+ 1.572 14
54	+ 0.592 84	+ 0.828 29	− 1.421 12	+ 2.147 02	54	+ 0.696 05	+ 0.788 84	− 1.484 88	+ 2.236 29
55	+ 1.037 24	+ 1.051 11	− 2.088 36	+ 3.165 03	55	+ 1.219 39	+ 0.973 68	− 2.193 08	+ 3.301 12
56	+ 2.076 20	+ 1.112 03	− 3.188 23	+ 4.817 35	56	+ 2.419 26	+ 0.943 90	− 3.363 18	+ 5.010 76
57	+ 4.561 11	+ 0.450 28	− 5.011 40	+ 7.344 32	57	+ 5.219 73	+ 0.077 57	− 5.297 30	+ 7.547 59
58	+10.154 89	− 2.290 72	− 7.864 16	+10.250 29	58	+11.281 01	− 3.005 64	− 8.275 35	+10.190 86
59	+19.856 15	− 8.398 13	−11.458 03	+10.295 66	59	+21.042 69	− 9.193 81	−11.848 89	+ 9.390 27

TABLE VII.—*Continued.*

VALUES OF A, B, C, AND D FOR THE ACTION OF VENUS ON THE MOON.

System 6.					System 7.				
i	*A*	*B*	*C*	*D*	*i*	*A*	*B*	*C*	*D*
0	+27.168 05	−13.660 15	−13.507 87	− 0.614 53	0	+26.709 86	−13.595 11	−13.114 76	− 2.251 50
1	+20.933 58	− 9.364 52	−11.569 06	− 8.839 67	1	+19.448 82	− 8.466 86	−10.981 95	− 9.682 40
2	+11.413 64	− 3.292 23	− 8.121 39	− 9.888 68	2	+10.203 37	− 2.547 15	− 7.656 24	− 9.922 08
3	+ 5.381 20	− 0.147 12	− 5.234 09	− 7.440 99	3	+ 4.702 14	+ 0.247 41	− 4.949 55	− 7.249 30
4	+ 2.541 06	+ 0.802 84	− 3.343 88	− 4.990 86	4	+ 2.189 26	+ 0.993 76	− 3.183 00	− 4.817 24
5	+ 1.301 93	+ 0.889 85	− 2.191 79	− 3.310 41	5	+ 1.112 88	+ 0.987 87	− 2.100 76	− 3.191 27
6	+ 0.751 59	+ 0.738 57	− 1.490 15	− 2.252 64	6	+ 0.642 27	+ 0.794 77	− 1.437 02	− 2.175 89
7	+ 0.493 04	+ 0.558 84	− 1.051 87	− 1.588 25	7	+ 0.424 78	+ 0.594 79	− 1.019 58	− 1.539 05
8	+ 0.362 49	+ 0.406 35	− 0.768 85	− 1.159 95	8	+ 0.316 91	+ 0.431 49	− 0.748 40	− 1.128 06
9	+ 0.291 15	+ 0.288 68	− 0.579 82	− 0.874 26	9	+ 0.258 97	+ 0.307 39	− 0.566 37	− 0.853 30
10	+ 0.248 83	+ 0.200 73	− 0.449 54	− 0.676 95	10	+ 0.225 10	+ 0.215 27	− 0.440 39	− 0.663 03
11	+ 0.221 64	+ 0.135 56	− 0.357 19	− 0.536 17	11	+ 0.203 54	+ 0.147 21	− 0.350 75	− 0.526 89
12	+ 0.202 85	+ 0.087 21	− 0.290 06	− 0.432 71	12	+ 0.188 67	+ 0.096 73	− 0.285 40	− 0.426 57
13	+ 0.189 02	+ 0.051 15	− 0.240 17	− 0.354 65	13	+ 0.177 68	+ 0.059 04	− 0.236 72	− 0.350 68
14	+ 0.178 33	+ 0.024 06	− 0.202 37	− 0.249 35	14	+ 0.169 10	+ 0.030 66	− 0.199 77	− 0.291 91
15	+ 0.169 72	+ 0.003 54	− 0.173 25	− 0.246 79	15	+ 0.162 13	+ 0.009 12	− 0.171 25	− 0.245 43
16	+ 0.162 59	− 0.012 10	− 0.150 49	− 0.208 55	16	+ 0.156 30	− 0.007 38	− 0.148 93	− 0.207 99
17	+ 0.156 58	− 0.024 12	− 0.132 48	− 0.177 28	17	+ 0.151 34	− 0.020 10	− 0.131 25	− 0.177 29
18	+ 0.151 46	− 0.033 39	− 0.118 07	− 0.151 29	18	+ 0.147 08	− 0.029 98	− 0.117 09	− 0.151 73
19	+ 0.147 05	− 0.040 59	− 0.106 46	− 0.129 37	19	+ 0.143 38	− 0.037 71	− 0.105 68	− 0.130 12
20	+ 0.143 24	− 0.046 21	− 0.097 03	− 0.110 61	20	+ 0.140 19	− 0.043 78	− 0.096 42	− 0.111 60
21	+ 0.139 95	− 0.050 60	− 0.089 36	− 0.094 35	21	+ 0.137 45	− 0.048 56	− 0.088 87	− 0.095 51
22	+ 0.137 13	− 0.054 04	− 0.083 10	− 0.080 05	22	+ 0.135 09	− 0.052 36	− 0.082 73	− 0.081 34
23	+ 0.134 72	− 0.056 72	− 0.078 00	− 0.067 31	23	+ 0.133 11	− 0.055 37	− 0.077 73	− 0.068 68
24	+ 0.132 70	− 0.058 81	− 0.073 88	− 0.055 81	24	+ 0.131 47	− 0.057 75	− 0.073 71	− 0.057 24
25	+ 0.131 04	− 0.060 43	− 0.070 60	− 0.045 28	25	+ 0.130 15	− 0.059 64	− 0.070 51	− 0.046 74
26	+ 0.129 70	− 0.061 66	− 0.068 04	− 0.035 51	26	+ 0.129 14	− 0.061 13	− 0.068 03	− 0.036 98
27	+ 0.128 69	− 0.062 57	− 0.066 12	− 0.026 31	27	+ 0.128 44	− 0.062 27	− 0.066 19	− 0.027 77
28	+ 0.127 99	− 0.063 20	− 0.064 78	− 0.017 52	28	+ 0.128 04	− 0.063 12	− 0.064 93	− 0.018 95
29	+ 0.127 58	− 0.063 59	− 0.064 00	− 0.008 99	29	+ 0.127 93	− 0.063 72	− 0.064 21	− 0.010 37
30	+ 0.127 47	− 0.063 75	− 0.063 73	− 0.000 59	30	+ 0.128 11	− 0.064 08	− 0.064 02	− 0.001 91
31	+ 0.127 66	− 0.063 68	− 0.063 97	+ 0.007 82	31	+ 0.128 53	− 0.064 23	− 0.064 36	+ 0.006 58
32	+ 0.128 14	− 0.063 40	− 0.064 75	+ 0.016 35	32	+ 0.129 37	− 0.064 14	− 0.065 23	+ 0.015 22
33	+ 0.128 93	− 0.062 87	− 0.066 06	+ 0.025 16	33	+ 0.130 46	− 0.063 82	− 0.066 66	+ 0.024 15
34	+ 0.130 03	− 0.062 07	− 0.067 96	+ 0.034 39	34	+ 0.131 88	− 0.063 20	− 0.068 67	+ 0.033 52
35	+ 0.131 46	− 0.060 95	− 0.070 52	+ 0.044 19	35	+ 0.133 65	− 0.062 28	− 0.071 36	+ 0.043 50
36	+ 0.133 25	− 0.059 43	− 0.073 80	+ 0.054 76	36	+ 0.135 77	− 0.060 97	− 0.074 81	+ 0.054 29
37	+ 0.135 38	− 0.057 46	− 0.077 93	+ 0.066 33	37	+ 0.138 31	− 0.059 19	− 0.079 12	+ 0.066 11
38	+ 0.137 94	− 0.054 88	− 0.083 05	+ 0.079 15	38	+ 0.141 28	− 0.056 83	− 0.084 45	+ 0.079 25
39	+ 0.140 92	− 0.051 56	− 0.089 35	+ 0.093 57	39	+ 0.144 74	− 0.053 72	− 0.091 01	+ 0.094 04
40	+ 0.144 39	− 0.047 29	− 0.097 11	+ 0.110 00	40	+ 0.148 74	− 0.049 66	− 0.099 08	+ 0.110 93
41	+ 0.148 43	− 0.041 77	− 0.106 65	+ 0.128 98	41	+ 0.153 39	− 0.044 38	− 0.109 00	+ 0.130 48
42	+ 0.153 10	− 0.034 67	− 0.118 44	+ 0.151 20	42	+ 0.158 77	− 0.037 50	− 0.121 27	+ 0.153 42
43	+ 0.158 55	− 0.025 46	− 0.133 11	+ 0.177 61	43	+ 0.165 05	− 0.028 53	− 0.136 53	+ 0.180 72
44	+ 0.164 95	− 0.013 47	− 0.151 49	+ 0.209 47	44	+ 0.172 46	− 0.016 77	− 0.155 67	+ 0.213 73
45	+ 0.172 57	+ 0.002 22	− 0.174 80	+ 0.248 52	45	+ 0.181 29	− 0.001 33	− 0.179 97	+ 0.254 26
46	+ 0.181 82	+ 0.022 90	− 0.204 72	+ 0.297 23	46	+ 0.192 06	+ 0.019 11	− 0.211 17	+ 0.304 93
47	+ 0.193 36	+ 0.050 33	− 0.243 70	+ 0.359 18	47	+ 0.205 57	+ 0.046 31	− 0.251 88	+ 0.369 51
48	+ 0.208 36	+ 0.087 00	− 0.295 37	+ 0.439 67	48	+ 0.223 17	+ 0.082 74	− 0.305 90	+ 0.453 58
49	+ 0.228 84	+ 0.136 37	− .365 20	+ 0.546 76	49	+ 0.247 21	+ 0.131 82	− 0.379 03	+ 0.565 69
50	+ 0.258 60	+ 0.203 17	− 0.461 77	+ 0.693 12	50	+ 0.282 11	+ 0.198 18	− 0.480 29	+ 0.719 23
51	+ 0.305 13	+ 0.293 60	− 0.598 74	+ 0.899 21	51	+ 0.336 42	+ 0.287 73	− 0.624 15	+ 0.935 91
52	+ 0.383 93	+ 0.414 79	− 0.798 70	+ 1.199 13	52	+ 0.427 73	+ 0.406 76	− 0.834 50	+ 1.251 84
53	+ 0.528 66	+ 0.571 44	− 1.100 11	+ 1.651 13	53	+ 0.594 08	+ 0.557 85	− 1.151 93	+ 1.728 56
54	+ 0.816 50	+ 0.753 57	− 1.570 07	+ 2.355 73	54	+ 0.921 98	+ 0.725 14	− 1.647 13	+ 2.471 57
55	+ 1.431 37	+ 0.895 83	− 2.327 19	+ 3.480 82	55	+ 1.616 38	+ 0.827 45	− 2.443 83	+ 3.653 74
56	+ 2.817 66	+ 0.757 50	− 3.575 16	+ 5.262 94	56	+ 3.166 15	+ 0.584 08	− 3.750 22	+ 5.503 98
57	+ 5.977 20	− 0.362 00	− 5.615 20	+ 7.807 13	57	+ 6.641 59	− 0.784 55	− 5.857 04	+ 8.051 79
58	+12.528 32	− 3.866 93	− 8.661 40	+10.107 54	58	+13.612 05	− 4.704 84	− 8.907 20	+10.018 16
59	+22.187 74	−10.133 34	−12.054 39	+ 8.285 33	59	+23.132 00	−11.051 60	−12.080 40	+ 7.240 47

TABLE VII.—*Continued.*

VALUES OF A, B, C, AND D FOR THE ACTION OF VENUS ON THE MOON.

System 8.					System 9.				
i	*A*	*B*	*C*	*D*	*i*	*A*	*B*	*C*	*D*
0	+27.214 80	−13.938 26	−13.276 55	− 3.500 11	0	+29.406 78	−14.968 87	−14.437 91	− 4.127 09
1	+19.039 87	− 8.020 33	−11.019 53	−10.665 09	1	+20.202 62	− 8.251 18	−11.951 44	−11.843 65
2	+ 9.631 65	− 1.990 26	− 7.641 36	−10.302 17	2	+ 9.902 42	− 1.721 59	− 8.180 82	−11.106 40
3	+ 4.310 63	+ 0.617 44	− 4.928 09	− 7.329 55	3	+ 4.288 72	− 0.911 18	− 5.199 88	− 7.719 82
4	+ 1.960 54	+ 1.206 21	− 3.166 75	− 4.809 35	4	+ 1.894 56	+ 1.406 55	− 3.301 10	− 4.984 16
5	+ 0.980 06	+ 1.109 85	− 2.089 90	− 3.167 93	5	+ 0.925 98	+ 1.232 53	− 2.158 52	− 3.248 67
6	+ 0.561 76	+ 0.868 13	− 1.429 89	− 2.155 37	6	+ 0.524 26	+ 0.942 42	− 1.466 67	− 2.195 56
7	+ 0.373 31	+ 0.641 49	− 1.014 79	− 1.524 13	7	+ 0.348 15	+ 0.687 36	− 1.035 53	− 1.546 04
8	+ 0.282 29	+ 0.462 81	− 0.745 09	− 1.117 93	8	+ 0.265 37	+ 0.491 96	− 0.757 34	− 1.131 04
9	+ 0.234 65	+ 0.329 36	− 0.564 01	− 0.846 69	9	+ 0.223 12	+ 0.348 43	− 0.571 56	− 0.855 24
10	+ 0.207 40	+ 0.231 26	− 0.438 66	− 0.658 89	10	+ 0.199 44	+ 0.244 06	− 0.443 51	− 0.664 92
11	+ 0.190 28	+ 0.159 18	− 0.349 46	− 0.524 46	11	+ 0.184 74	+ 0.167 97	− 0.352 70	− 0.528 99
12	+ 0.178 51	+ 0.105 91	− 0.284 42	− 0.425 32	12	+ 0.174 64	+ 0.112 05	− 0.286 67	− 0.428 89
13	+ 0.169 76	+ 0.066 21	− 0.235 97	− 0.350 25	13	+ 0.167 08	+ 0.070 53	− 0.237 61	− 0.353 18
14	+ 0.162 85	+ 0.036 33	− 0.199 20	− 0.292 05	14	+ 0.161 05	+ 0.039 40	− 0.200 43	− 0.294 52
15	+ 0.157 17	+ 0.013 65	− 0.170 81	− 0.245 98	15	+ 0.155 99	+ 0.015 80	− 0.171 81	− 0.248 10
16	+ 0.152 34	− 0.003 74	− 0.148 61	− 0.208 80	16	+ 0.151 67	− 0.002 23	− 0.149 43	− 0.210 65
17	+ 0.148 19	− 0.017 17	− 0.131 02	− 0.178 30	17	+ 0.147 90	− 0.016 16	− 0.131 73	− 0.179 93
18	+ 0.144 58	− 0.027 63	− 0.116 96	− 0.152 86	18	+ 0.144 58	− 0.027 01	− 0.117 60	− 0.154 30
19	+ 0.141 43	− 0.035 83	− 0.105 61	− 0.131 33	19	+ 0.141 69	− 0.035 49	− 0.106 22	− 0.132 62
20	+ 0.138 71	− 0.042 29	− 0.096 41	− 0.112 85	20	+ 0.139 16	− 0.042 17	− 0.097 01	− 0.114 00
21	+ 0.136 36	− 0.047 43	− 0.088 94	− 0.096 78	21	+ 0.136 98	− 0.047 47	− 0.089 52	− 0.097 79
22	+ 0.134 36	− 0.051 51	− 0.082 85	− 0.082 60	22	+ 0.135 12	− 0.051 70	− 0.083 43	− 0.083 50
23	+ 0.132 68	− 0.054 78	− 0.077 92	− 0.069 93	23	+ 0.133 57	− 0.055 08	− 0.078 50	− 0.070 72
24	+ 0.131 33	− 0.057 38	− 0.073 94	− 0.058 46	24	+ 0.132 33	− 0.057 79	− 0.074 54	− 0.059 14
25	+ 0.130 27	− 0.059 48	− 0.070 80	− 0.047 91	25	+ 0.131 37	− 0.059 96	− 0.071 41	− 0.048 50
26	+ 0.129 51	− 0.061 14	− 0.068 37	− 0.038 10	26	+ 0.130 70	− 0.061 69	− 0.069 01	− 0.038 58
27	+ 0.129 04	− 0.062 45	− 0.066 59	− 0.028 82	27	+ 0.130 31	− 0.063 05	− 0.067 25	− 0.029 21
28	+ 0.128 85	− 0.063 46	− 0.065 39	− 0.019 93	28	+ 0.130 20	− 0.064 12	− 0.066 08	− 0.020 21
29	+ 0.128 94	− 0.064 21	− 0.064 75	− 0.011 27	29	+ 0.130 37	− 0.064 90	− 0.065 46	− 0.011 45
30	+ 0.129 33	− 0.064 71	− 0.064 63	− 0.002 71	30	+ 0.130 83	− 0.065 44	− 0.065 38	− 0.002 79
31	+ 0.130 02	− 0.064 98	− 0.065 04	+ 0.005 89	31	+ 0.131 59	− 0.065 74	− 0.065 83	+ 0.005 91
32	+ 0.131 01	− 0.065 02	− 0.065 99	+ 0.014 65	32	+ 0.132 63	− 0.065 82	− 0.066 83	+ 0.014 78
33	+ 0.132 32	− 0.064 82	− 0.067 51	+ 0.023 72	33	+ 0.134 01	− 0.065 63	− 0.068 39	+ 0.023 97
34	+ 0.133 97	− 0.064 32	− 0.069 63	+ 0.033 25	34	+ 0.135 71	− 0.065 15	− 0.070 57	+ 0.033 62
35	+ 0.135 96	− 0.063 52	− 0.072 44	+ 0.043 41	35	+ 0.137 77	− 0.064 35	− 0.073 43	+ 0.043 92
36	+ 0.138 34	− 0.062 33	− 0.076 02	+ 0.054 41	36	+ 0.140 22	− 0.063 15	− 0.077 06	+ 0.055 06
37	+ 0.141 13	− 0.060 66	− 0.080 48	+ 0.066 47	37	+ 0.143 08	− 0.061 47	− 0.081 59	+ 0.067 28
38	+ 0.144 40	− 0.058 40	− 0.085 99	+ 0.079 89	38	+ 0.146 40	− 0.059 20	− 0.087 18	+ 0.080 87
39	+ 0.148 16	− 0.055 41	− 0.092 77	+ 0.095 01	39	+ 0.150 22	− 0.056 18	− 0.094 05	+ 0.096 19
40	+ 0.152 52	− 0.051 45	− 0.101 08	+ 0.112 29	40	+ 0.154 64	− 0.052 18	− 0.102 46	+ 0.113 69
41	+ 0.157 56	− 0.046 26	− 0.111 29	+ 0.132 31	41	+ 0.159 73	− 0.046 94	− 0.112 79	+ 0.133 96
42	+ 0.163 40	− 0.039 47	− 0.123 92	+ 0.155 82	42	+ 0.165 61	− 0.040 07	− 0.125 54	+ 0.157 76
43	+ 0.170 20	− 0.030 58	− 0.139 61	+ 0.183 84	43	+ 0.172 45	− 0.031 06	− 0.141 39	+ 0.186 11
44	+ 0.178 22	− 0.018 89	− 0.159 31	+ 0.217 72	44	+ 0.180 48	− 0.019 21	− 0.161 26	+ 0.220 39
45	+ 0.187 79	− 0.003 48	− 0.184 30	+ 0.259 39	45	+ 0.190 04	− 0.003 58	− 0.186 46	+ 0.262 54
46	+ 0.199 46	+ 0.016 95	− 0.216 40	+ 0.311 52	46	+ 0.201 66	+ 0.017 16	− 0.218 84	+ 0.315 27
47	+ 0.214 10	+ 0.044 19	− 0.258 28	+ 0.378 02	47	+ 0.216 18	+ 0.044 88	− 0.261 04	+ 0.382 55
48	+ 0.233 13	+ 0.080 75	− 0.313 87	+ 0.464 69	48	+ 0.234 99	+ 0.082 10	− 0.317 09	+ 0.470 25
49	+ 0.259 12	+ 0.130 04	− 0.389 15	+ 0.580 41	49	+ 0.260 56	+ 0.132 43	− 0.392 99	+ 0.587 39
50	+ 0.296 74	+ 0.196 72	− 0.493 44	+ 0.739 12	50	+ 0.297 46	+ 0.200 73	− 0.498 20	+ 0.748 16
51	+ 0.355 09	+ 0.286 62	− 0.641 69	+ 0.963 40	51	+ 0.354 63	+ 0.293 26	− 0.647 89	+ 0.975 66
52	+ 0.452 86	+ 0.405 76	− 0.858 63	+ 1.200 97	52	+ 0.450 50	+ 0.416 79	− 0.867 29	+ 1.308 62
53	+ 0.630 50	+ 0.555 78	− 1.186 20	+ 1.786 19	53	+ 0.625 31	+ 0.574 26	− 1.199 57	+ 1.813 71
54	+ 0.980 18	+ 0.717 66	− 1.697 83	+ 2.559 44	54	+ 0.971 84	+ 0.748 81	− 1.720 64	+ 2.606 76
55	+ 1.720 92	+ 0.800 56	− 2.521 46	+ 3.791 24	55	+ 1.713 05	+ 0.851 55	− 2.565 51	+ 3.881 50
56	+ 3.376 87	+ 0.495 10	− 3.871 98	+ 5.716 19	56	+ 3.399 86	+ 0.566 78	− 3.966 64	+ 5.903 18
57	+ 7.094 11	− 1.048 90	− 6.045 21	+ 8.332 90	57	+ 7.271 80	− 1.008 83	− 6.262 96	+ 8.721 65
58	+14.504 38	− 5.336 43	− 9.167 95	+10.174 19	58	+15.228 32	− 5.570 39	− 9.657 91	+10.830 84
59	+24.307 55	−11.957 39	−12.350 17	+ 6.726 53	59	+26.106 78	−12.841 66	−13.265 09	+ 7.187 45

TABLE VII.—*Concluded.*

VALUES OF A, B, C, AND D FOR THE ACTION OF VENUS ON THE MOON.

System 10.					System 11.				
i	*A*	*B*	*C*	*D*	*i*	*A*	*B*	*C*	*D*
0	+32.969 68	−16.569 97	−16.399 71	− 3.668 44	0	+35.831 31	−17.896 16	−17.935 12	− 1.712 41
1	+22.878 40	− 9.298 85	−13.579 54	−12.862 57	1	+25.980 88	−10.969 56	−15.011 32	−12.871 70
2	+11.019 61	− 1.898 25	− 9.121 36	−12.148 82	2	+12.649 60	− 2.655 75	− 9.993 85	−12.897 08
3	+ 4.649 91	− 1.018 42	− 5.668 34	− 8.326 79	3	+ 5.312 22	− 0.803 15	− 6.115 36	− 8.902 19
4	+ 2.004 91	+ 1.525 68	− 3.530 58	− 5.294 55	4	+ 2.274 38	+ 1.480 38	− 3.754 76	− 5.628 18
5	+ 0.961 50	+ 1.313 85	− 2.275 35	− 3.408 98	5	+ 1.084 48	+ 1.308 02	− 2.392 50	− 3.595 21
6	+ 0.537 67	+ 0.991 88	− 1.529 55	− 2.283 06	6	+ 0.602 79	+ 0.991 77	− 1.594 55	− 2.390 23
7	+ 0.354 96	+ 0.716 40	− 1.071 36	− 1.596 99	7	+ 0.394 57	+ 0.715 16	− 1.109 73	− 1.661 64
8	+ 0.270 14	+ 0.508 75	− 0.778 90	− 1.162 57	8	+ 0.297 06	+ 0.505 83	− 0.802 90	− 1.203 41
9	+ 0.227 24	+ 0.357 97	− 0.585 20	− 0.875 82	9	+ 0.247 04	+ 0.353 97	− 0.601 02	− 0.902 67
10	+ 0.203 27	+ 0.249 27	− 0.452 55	− 0.678 94	10	+ 0.218 68	+ 0.244 78	− 0.463 47	− 0.697 18
11	+ 0.188 39	+ 0.170 58	− 0.358 96	− 0.538 89	11	+ 0.200 85	+ 0.165 98	− 0.366 83	− 0.551 59
12	+ 0.178 13	+ 0.113 07	− 0.291 19	− 0.436 09	12	+ 0.188 48	+ 0.108 59	− 0.297 07	− 0.445 09
13	+ 0.170 40	+ 0.070 59	− 0.241 00	− 0.358 52	13	+ 0.179 19	+ 0.066 36	− 0.245 53	− 0.364 97
14	+ 0.164 20	+ 0.038 87	− 0.203 07	− 0.298 55	14	+ 0.171 75	+ 0.034 92	− 0.206 08	− 0.303 18
15	+ 0.159 00	+ 0.014 93	− 0.173 92	− 0.251 17	15	+ 0.165 57	+ 0.011 29	− 0.176 86	− 0.254 49
16	+ 0.154 52	− 0.003 31	− 0.151 19	− 0.213 01	16	+ 0.160 28	− 0.006 65	− 0.153 64	− 0.215 35
17	+ 0.150 59	− 0.017 36	− 0.133 25	− 0.181 73	17	+ 0.155 70	− 0.020 38	− 0.135 31	− 0.183 33
18	+ 0.147 15	− 0.028 23	− 0.118 92	− 0.155 68	18	+ 0.151 69	− 0.030 98	− 0.120 70	− 0.156 70
19	+ 0.144 13	− 0.036 72	− 0.107 39	− 0.133 65	19	+ 0.148 17	− 0.039 22	− 0.108 96	− 0.134 21
20	+ 0.141 48	− 0.043 40	− 0.098 08	− 0.114 75	20	+ 0.145 09	− 0.045 66	− 0.099 45	− 0.114 95
21	+ 0.139 19	− 0.048 67	− 0.090 50	− 0.098 31	21	+ 0.142 43	− 0.050 70	− 0.091 72	− 0.098 23
22	+ 0.137 22	− 0.052 86	− 0.084 36	− 0.083 81	22	+ 0.140 12	− 0.054 68	− 0.085 45	− 0.083 50
23	+ 0.135 57	− 0.056 20	− 0.079 38	− 0.070 86	23	+ 0.138 16	− 0.057 81	− 0.080 36	− 0.070 35
24	+ 0.134 22	− 0.058 85	− 0.075 38	− 0.059 13	24	+ 0.136 53	− 0.060 27	− 0.076 26	− 0.058 46
25	+ 0.133 17	− 0.060 96	− 0.072 22	− 0.048 36	25	+ 0.135 22	− 0.062 20	− 0.073 01	− 0.047 55
26	+ 0.132 41	− 0.062 63	− 0.069 79	− 0.038 32	26	+ 0.134 20	− 0.063 70	− 0.070 50	− 0.037 41
27	+ 0.131 93	− 0.063 93	− 0.068 01	− 0.028 84	27	+ 0.133 48	− 0.064 83	− 0.068 64	− 0.027 83
28	+ 0.131 73	− 0.064 92	− 0.066 81	− 0.019 75	28	+ 0.133 05	− 0.065 65	− 0.067 38	− 0.018 66
29	+ 0.131 82	− 0.065 63	− 0.066 18	− 0.010 90	29	+ 0.132 89	− 0.066 20	− 0.066 68	− 0.009 75
30	+ 0.132 18	− 0.066 10	− 0.066 08	− 0.002 15	30	+ 0.133 02	− 0.066 51	− 0.066 51	− 0.000 96
31	+ 0.132 84	− 0.066 32	− 0.066 51	+ 0.006 62	31	+ 0.133 44	− 0.066 56	− 0.066 87	+ 0.007 85
32	+ 0.133 79	− 0.066 31	− 0.067 49	+ 0.015 57	32	+ 0.134 15	− 0.066 37	− 0.067 77	+ 0.016 82
33	+ 0.135 06	− 0.066 03	− 0.069 03	+ 0.024 82	33	+ 0.135 15	− 0.065 91	− 0.069 23	+ 0.026 08
34	+ 0.136 64	− 0.065 45	− 0.071 18	+ 0.034 54	34	+ 0.136 46	− 0.065 16	− 0.071 30	+ 0.035 78
35	+ 0.138 57	− 0.064 54	− 0.074 02	+ 0.044 89	35	+ 0.138 10	− 0.064 05	− 0.074 04	+ 0.046 11
36	+ 0.140 86	− 0.063 24	− 0.077 63	+ 0.056 08	36	+ 0.140 07	− 0.062 54	− 0.077 55	+ 0.057 24
37	+ 0.143 55	− 0.061 43	− 0.082 13	+ 0.068 34	37	+ 0.142 42	− 0.060 50	− 0.081 92	+ 0.069 42
38	+ 0.146 68	− 0.059 01	− 0.087 67	+ 0.081 96	38	+ 0.145 15	− 0.057 84	− 0.087 32	+ 0.082 93
39	+ 0.150 29	− 0.055 81	− 0.094 47	+ 0.097 30	39	+ 0.148 32	− 0.054 37	− 0.093 95	+ 0.098 11
40	+ 0.154 44	− 0.051 63	− 0.102 80	+ 0.114 80	40	+ 0.151 96	− 0.049 87	− 0.102 09	+ 0.115 40
41	+ 0.159 20	− 0.046 17	− 0.113 05	+ 0.135 04	41	+ 0.156 14	− 0.044 06	− 0.112 08	+ 0.135 36
42	+ 0.164 70	− 0.039 03	− 0.125 68	+ 0.158 78	42	+ 0.160 93	− 0.036 52	− 0.124 42	+ 0.158 73
43	+ 0.171 06	− 0.029 69	− 0.141 37	+ 0.187 03	43	+ 0.166 46	− 0.026 71	− 0.139 73	+ 0.186 47
44	+ 0.178 50	− 0.017 45	− 0.161 05	+ 0.221 16	44	+ 0.172 85	− 0.013 91	− 0.158 94	+ 0.219 93
45	+ 0.187 29	− 0.001 31	− 0.185 99	+ 0.263 06	45	+ 0.180 36	+ 0.002 93	− 0.183 28	+ 0.260 93
46	+ 0.197 90	+ 0.020 12	− 0.218 01	+ 0.315 43	46	+ 0.189 32	+ 0.025 21	− 0.214 53	+ 0.312 08
47	+ 0.211 05	+ 0.048 73	− 0.259 78	+ 0.382 19	47	+ 0.200 31	+ 0.054 97	− 0.255 27	+ 0.377 15
48	+ 0.227 98	+ 0.087 26	− 0.315 23	+ 0.469 15	48	+ 0.214 31	+ 0.095 04	− 0.309 35	+ 0.461 76
49	+ 0.250 86	+ 0.139 52	− 0.390 37	+ 0.585 26	49	+ 0.233 09	+ 0.149 53	− 0.382 62	+ 0.574 53
50	+ 0.283 78	+ 0.210 84	− 0.494 61	+ 0.744 61	50	+ 0.260 04	+ 0.224 24	− 0.484 27	+ 0.729 10
51	+ 0.334 81	+ 0.308 36	− 0.643 17	+ 0.970 29	51	+ 0.301 96	+ 0.327 27	− 0.629 23	+ 0.947 79
52	+ 0.420 84	+ 0.440 62	− 0.861 46	+ 1.301 21	52	+ 0.373 30	+ 0.469 14	− 0.842 44	+ 1.268 49
53	+ 0.579 24	+ 0.614 18	− 1.193 43	+ 1.805 20	53	+ 0.506 61	+ 0.660 75	− 1.167 37	+ 1.757 69
54	+ 0.897 75	+ 0.819 82	− 1.717 55	+ 2.602 39	54	+ 0.779 60	+ 0.902 68	− 1.682 27	+ 2.535 09
55	+ 1.592 91	+ 0.983 93	− 2.576 84	+ 3.901 43	55	+ 1.388 11	+ 1.144 02	− 2.532 13	+ 3.815 21
56	+ 3.213 60	+ 0.814 77	− 4.028 36	+ 6.015 62	56	+ 2.843 29	+ 1.141 04	− 3.985 24	+ 5.946 55
57	+ 7.077 92	− 0.596 54	− 6.481 38	+ 9.125 85	57	+ 6.437 87	+ 0.057 82	− 6.495 70	+ 9.252 71
58	+15.483 72	− 5.189 93	−10.293 80	+11.905 97	58	+14.717 82	− 4.162 63	−10.555 20	+12.780 88
59	+28.021 10	−13.352 48	−14.668 60	+ 8.801 20	59	+28.397 74	−12.848 84	−15.548 89	+10.996 73

TABLE VIII.

Development of A, B, C, and D for Venus in periodic Series.

Coeff. of v, *g′*	*A*		*B*		*C*		*D*	
	cos	sin	cos	sin	cos	sin	cos	sin
0 0	+2.1947	0.0000	−0.5886	0.0000	−1.6061	0.0000	−0.0005	0.0000
0+ 1	+0.1742	−0.0428	−0.0314	+0.0085	−0.1428	+0.0343	+0.0080	+0.0774
0+ 2	+0.0523	−0.0556	−0.0086	+0.0084	−0.0438	+0.0471	+0.0124	+0.0131
0+ 3	+0.0046	−0.0075	−0.0008	+0.0012	−0.0038	+0.0063	+0.0022	+0.0016
0+ 4	+0.0001	−0.0015	0.0000	0.0000	−0.0001	+0.0012	+0.0004	0.0000
0+ 5	−0.0001	−0.0002	0.0000	+0.0002	0.0000	+0.0002	0.0000	0.0000
0+ 6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1− 7	0.0000	0.0000	−0.0001	+0.0001	0.0000	0.0000	+0.0001	0.0000
1− 6	−0.0004	+0.0003	−0.0003	0.0000	+0.0001	−0.0003	0.0000	−0.0001
1− 5	+0.0004	+0.0014	−0.0002	−0.0005	−0.0001	−0.0010	+0.0005	−0.0005
1− 4	+0.0049	+0.0084	−0.0013	−0.0017	−0.0038	−0.0068	+0.0031	−0.0021
1− 3	+0.0546	+0.0552	−0.0127	−0.0115	−0.0423	−0.0441	+0.0182	−0.0199
1− 2	+0.2534	+0.0552	−0.0820	−0.0156	−0.1708	−0.0393	+0.0201	−0.1214
1− 1	+4.0006	−0.0006	−1.1590	+0.0004	−2.8416	0.0000	−0.0015	−1.1602
1+ 0	+0.0924	−0.0288	+0.0118	+0.0030	−0.1036	+0.0255	−0.0010	+0.0525
1+ 1	+0.0501	−0.0549	−0.0058	+0.0060	−0.0445	+0.0490	+0.0063	+0.0068
1+ 2	+0.0038	−0.0064	−0.0002	+0.0008	−0.0036	+0.0059	+0.0013	+0.0009
1+ 3	0.0000	−0.0014	0.0000	+0.0005	0.0000	+0.0013	+0.0004	+0.0001
1+ 4	−0.0001	−0.0002	0.0000	0.0000	+0.0002	+0.0004	−0.0002	0.0000
1+ 5	0.0000	0.0000	−0.0001	+0.0001	0.0000	0.0000	+0.0001	0.0000
2− 8	0.0000	0.0000	−0.0001	−0.0001	+0.0001	0.0000	+0.0001	0.0000
2− 7	0.0000	+0.0004	−0.0002	−0.0001	+0.0001	−0.0006	+0.0002	0.0000
2− 6	+0.0004	+0.0015	0.0000	−0.0005	−0.0002	−0.0011	+0.0002	−0.0001
2− 5	+0.0054	+0.0092	−0.0016	−0.0025	−0.0036	−0.0065	+0.0037	−0.0027
2− 4	+0.0567	+0.0542	−0.0174	−0.0144	−0.0398	−0.0398	+0.0223	−0.0252
2− 3	+0.3093	+0.0648	−0.1281	−0.0245	−0.1826	−0.0402	+0.0315	−0.1729
2− 2	+3.7105	−0.0014	−1.3562	+0.0011	−2.3535	0.0000	−0.0018	−1.6716
2− 1	+0.0402	−0.0173	+0.0321	+0.0010	−0.0730	+0.0162	−0.0016	+0.0516
2+ 0	+0.0483	−0.0536	−0.0049	+0.0052	−0.0430	+0.0486	+0.0001	+0.0012
2+ 1	+0.0032	−0.0054	−0.0003	+0.0003	−0.0028	+0.0048	+0.0008	+0.0007
2+ 2	+0.0001	−0.0010	0.0000	+0.0002	.00000	+0.0011	+0.0001	0.0000
2+ 3	−0.0002	−0.0002	−0.0003	−0.0001	+0.0003	−0.0002	+0.0002	0.0000
2+ 4	0.0000	0.0000	−0.0001	−0.0001	+0.0001	0.0000	+0.0001	0.0000
3− 9	−0.0001	+0.0001	−0.0001	0.0000	+0.0001	0.0000	+0.0001	+0.0001
3− 8	0.0000	+0.0006	0.0000	−0.0003	+0.0001	−0.0003	+0.0001	0.0000
3− 7	+0.0004	+0.0020	+0.0001	−0.0007	−0.0003	−0.0011	+0.0007	−0.0002
3− 6	+0.0062	+0.0093	−0.0020	−0.0032	−0.0041	−0.0064	+0.0046	−0.0033
3− 5	+0.0586	+0.0524	−0.0217	−0.0168	−0.0366	−0.0354	+0.0252	−0.0305
3− 4	+0.3454	+0.0705	−0.1657	−0.0317	−0.1789	−0.0387	+0.0408	−0.2154
3− 3	+3.3553	−0.0017	−1.4697	+0.0012	−1.8858	+0.0003	−0.0014	−1.8630
3− 2	+0.0042	−0.0079	+0.0456	−0.0014	−0.0490	+0.0094	+0.0008	+0.0547
3− 1	+0.0454	−0.0516	−0.0044	+0.0049	−0.0410	+0.0463	−0.0056	−0.0043
3− 0	+0.0030	−0.0045	−0.0002	0.0000	−0.0030	+0.0044	0.0000	+0.0003
3− 1	−0.0003	−0.0010	+0.0004	+0.0002	+0.0001	+0.0011	+0.0002	0.0000
3− 2	−0.0004	0.0000	−0.0002	−0.0001	+.00002	−0.0002	0.0000	+0.0001
3− 3	−0.0001	+0.0001	−0.0001	0.0000	+0.0001	0.0000	+0.0001	+0.0001
4−10	−0.0001	+0.0002	+0.0001	+0.0001	+0.0002	0.0000	+0.0001	0.0000
4− 9	−0.0001	+0.0004	+0.0003	+0.0001	0.0000	−0.0002	0.0000	−0.0002
4− 8	+0.0003	+0.0018	0.0000	−0.0008	−0.0001	−0.0014	+0.0013	−0.0003
4− 7	+0.0064	+0.0096	−0.0025	−0.0036	−0.0038	−0.0060	+0.0052	−0.0036
4− 6	+0.0584	+0.0498	−0.0252	−0.0190	−0.0332	−0.0308	+0.0267	−0.0344
4− 5	+0.3605	+0.0726	−0.1938	−0.0370	−0.1671	−0.0355	+0.0468	−0.2442
4− 4	+2.9597	−0.0013	−1.4783	+0.0012	−1.4817	+0.0002	−0.0014	−1.8596
4− 3	−0.0206	−0.0005	+0.0526	−0.0043	−0.0330	+0.0045	+0.0037	+0.0580
4− 2	+0.0424	−0.0483	−0.0058	+0.0059	−0.0369	+0.0420	−0.0090	−0.0075
4− 1	+0.0023	−0.0034	0.0000	−0.0001	−0.0025	+0.0034	0.0000	+0.0003

TABLE VIII.—*Concluded.*

DEVELOPMENT OF A, B, C, AND D FOR VENUS IN PERIODIC SERIES.

Coeff. of v, *g′*	*A*		*B*		*C*		*D*	
	cos	sin	cos	sin	cos	sin	cos	sin
4+ 0	−0.0002	−0.0010	+0.0001	+0.0001	+0.0002	+0.0013	−0.0002	−0.0003
4+ 1	0.0000	+0.0002	−0.0001	−0.0001	0.0000	+0.0003	−0.0001	0.0000
4+ 2	+0.0001	+0.0002	−0.0001	+0.0001	+0.0004	0.0000	+0.0001	0.0000
5−11	0.0000	0.0000	−0.0001	0.0000	+0.0001	−0.0001	0.0000	−0.0001
5−10	−0.0004	0.0000	−0.0001	−0.0004	+0.0002	+0.0001	+0.0002	0.0000
5− 9	+0.0004	+0.0018	−0.0001	−0.0008	0.0000	−0.0011	+0.0011	−0.0003
5− 8	+0.0063	+0.0095	−0.0028	−0.0036	−0.0037	−0.0056	+0.0057	−0.0040
5− 7	+0.0579	+0.0465	−0.0284	−0.0200	−0.0298	−0.0208	+0.0276	−0.0374
5− 6	+0.3602	+0.0712	−0.2094	−0.0400	−0.1501	−0.0313	+0.0497	−0.2587
5− 5	+2.5539	−0.0014	−1.4070	+0.0012	−1.1407	+0.0002	−0.0015	−1.7422
5− 4	−0.0350	+0.0047	+0.0564	−0.0068	−0.0205	+0.0016	+0.0067	+0.0598
5− 3	+0.0389	−0.0445	−0.0064	+0.0069	−0.0328	+0.0375	−0.0122	−0.0108
5− 2	+0.0018	−0.0028	+0.0001	+0.0001	−0.0018	+0.0029	+0.0002	+0.0002
5− 1	+0.0004	−0.0011	0.0000	+0.0002	+0.0002	+0.0012	−0.0002	0.0000
5+ 0	−0.0001	+0.0002	−0.0002	+0.0002	−0.0001	+0.0001	+0.0002	0.0000
5+ 1	0.0000	0.0000	−0.0001	0.0000	+0.0001	−0.0001	0.0000	−0.0001
6−12	−0.0001	0.0000	0.0000	0.0000	+0.0002	0.0000	+0.0001	−0.0002
6−11	−0.0001	+0.0003	+0.0002	0.0000	+0.0004	+0.0001	+0.0002	0.0000
6−10	+0.0003	+0.0018	−0.0002	−0.0008	−0.0005	−0.0009	+0.0012	−0.0005
6− 9	+0.0061	+0.0091	−0.0035	−0.0038	−0.0033	−0.0050	+0.0060	−0.0040
6− 8	+0.0557	+0.0428	−0.0295	−0.0203	−0.0264	−0.0225	+0.0274	−0.0380
6− 7	+0.3455	+0.0684	−0.2150	−0.0414	−0.1312	−0.0274	+0.0500	−0.2590
6− 6	+2.1645	−0.0007	−1.2859	+0.0009	−0.8788	−0.0001	−0.0009	−1.5648
6− 5	−0.0436	+0.0086	+0.0558	−0.0084	−0.0128	+0.0002	+0.0086	+0.0588
6− 4	+0.0356	−0.0403	−0.0075	+0.0078	−0.0282	+0.0321	−0.0139	−0.0120
6− 3	+0.0016	−0.0022	0.0000	−0.0001	−0.0018	+0.0021	+0.0002	−0.0001
6− 2	0.0000	−0.0009	+0.0002	+0.0002	−0.0005	+0.0009	0.0000	+0.0005
6− 1	−0.0001	+0.0002	−0.0002	+0.0001	−0.0001	+0.0001	−0.0002	−0.0006
6+ 0	−0.0001	0.0000	0.0000	0.0000	−0.0002	0.0000	−0.0001	−0.0002
7−13	−0.0001	0.0000	0.0000	−0.0001	−0.0001	0.0000	−0.0002	0.0000
7−12	0.0000	+0.0005	+0.0003	−0.0001	−0.0001	−0.0001	+0.0004	+0.0002
7−11	+0.0006	+0.0017	+0.0001	−0.0009	−0.0002	−0.0009	+0.0007	−0.0001
7−10	+0.0064	+0.0088	−0.0033	−0.0042	−0.0030	−0.0046	+0.0058	−0.0043
7− 9	+0.0532	+0.0392	−0.0305	−0.0200	−0.0223	−0.0184	+0.0260	−0.0386
7− 8	+0.3227	+0.0632	−0.2112	−0.0407	−0.1116	−0.0226	+0.0486	−0.2508
7− 7	+1.8074	−0.0006	−1.1384	+0.0008	−0.6689	−0.0002	−0.0009	−1.3643
7− 6	−0.0458	+0.0106	−0.0547	−0.0102	−0.0074	−0.0009	+0.0105	+0.0567
7− 5	+0.0315	−0.0359	−0.0074	+0.0082	−0.0240	+0.0274	−0.0148	−0.0129
7− 4	+0.0012	−0.0015	+0.0001	−0.0004	−0.0012	+0.0016	+0.0002	+0.0002
7− 3	0.0000	−0.0011	+0.0003	+0.0003	0.0000	+0.0008	−0.0005	+0.0002
7− 2	−0.0001	−0.0002	−0.0001	+0.0002	0.0000	+0.0001	−0.0001	0.0000
7− 1	−0.0001	0.0000	0.0000	−0.0002	−0.0001	0.0000	−0.0002	0.0000
8−14	−0.0001	0.0000	0.0000	−0.0001	0.0000	−0.0002	+0.0001	0.0000
8−13	−0.0002	+0.0002	+0.0002	−0.0002	0.0000	−0.0001	+0.0002	−0.0003
8−12	+0.0002	+0.0014	−0.0002	−0.0008	−0.0002	−0.0010	+0.0013	−0.0006
8−11	+0.0058	+0.0080	−0.0034	−0.0044	−0.0027	−0.0037	+0.0050	−0.0041
8−10	+0.0496	+0.0351	−0.0306	−0.0190	−0.0193	−0.0155	+0.0250	−0.0374
8− 9	+0.2941	+0.0577	−0.1996	−0.0385	−0.0942	−0.0192	+0.0459	−0.2335
8− 8	+1.4892	−0.0001	−0.9842	+0.0003	−0.5050	−0.0001	−0.0006	−1.1634
8− 7	−0.0461	+0.0118	+0.0500	−0.0104	−0.0048	−0.0012	+0.0115	+0.0508
8− 6	+0.0278	−0.0316	−0.0081	+0.0085	−0.0168	+0.0228	−0.0144	−0.0122
8− 5	+0.0010	−0.0010	+0.0001	−0.0004	−0.0012	+0.0012	−0.0005	0.0000
8− 4	−0.0001	−0.0009	0.0000	−0.0001	−0.0002	+0.0009	+0.0001	+0.0002
8− 3	+0.0002	0.0000	0.0000	−0.0002	−0.0001	+0.0002	−0.0001	+0.0005
8− 2	−0.0001	0.0000	0.0000	−0.0001	0.0000	−0.0002	+0.0001	0.0000

TABLE IX.

COMPUTATION OF THE COEFFICIENTS FOR THE HANSENIAN VENUS-TERM OF LONG PERIOD.

System.	A_0''	$A_{c,1}$	$A_{c,2}$	$A_{s,1}$	$A_{s,2}$
0	+33.3948	−48.0672	−15.2486	+ 1.6810	+ 1.6959
1	+30.4226	−44.3154	−14.3087	+ 5.2845	+ 5.6929
2	+27.3802	−40.4460	−13.3384	+ 6.9388	+ 7.7952
3	+25.7695	−38.3437	−12.7915	+ 6.7824	+ 7.8308
4	+25.3715	−37.7570	−12.6090	+ 5.0267	+ 5.8930
5	+25.2326	−37.5000	−12.5055	+ 1.9933	+ 2.3924
6	+24.7525	−36.8595	−12.3300	− 1.4599	− 1.6609
7	+24.2976	−36.3246	−12.2190	− 4.3516	− 5.1130
8	+24.8273	−37.1237	−12.4835	− 6.2219	− 7.2996
9	+27.0732	−40.1587	−13.3322	− 6.9149	− 7.9252
10	+30.7124	−44.8651	−14.5505	− 5.9534	− 6.5819
11	+33.6483	−48.4814	−15.4035	− 2.7899	− 3.0391
	B_0''	$B_{c,1}$	$B_{c,2}$	$B_{s,1}$	$B_{s,2}$
0	−19.9136	+27.3868	+ 7.9770	− 1.3391	− 1.2601
1	−18.5130	+25.9046	+ 7.7612	− 4.0210	− 4.0255
2	−16.9031	+24.1227	+ 7.4655	− 5.0839	− 5.2866
3	−15.8666	+22.8730	+ 7.2028	− 4.7701	− 5.0902
4	−15.4249	+22.2499	+ 7.0232	− 3.4567	− 3.7608
5	−15.2448	+21.9597	+ 6.9235	− 1.4491	− 1.6516
6	−15.0836	+21.7873	+ 6.9008	+ 0.8330	+ 0.8101
7	−15.0404	+21.8370	+ 6.9698	+ 2.9218	+ 3.1445
8	−15.4663	+22.4592	+ 7.1627	+ 4.4869	+ 4.9005
9	−16.6559	+23.9194	+ 7.4839	+ 5.2206	+ 5.6178
10	−18.4569	+25.9344	+ 7.8260	+ 4.5771	+ 4.7687
11	−19.9292	+27.4474	+ 8.0139	+ 2.1201	+ 2.1782
	C_0''	$C_{c,1}$	$C_{c,2}$	$C_{s,1}$	$C_{s,2}$
0	−13.4811	+20.6804	+ 7.2716	− 0.3420	− 0.4358
1	−11.9096	+18.4107	+ 6.5475	− 1.2636	− 1.6673
2	−10.4771	+16.3233	+ 5.8729	− 1.8550	− 2.0586
3	− 9.9030	+15.4707	+ 5.5887	− 2.0122	− 2.7407
4	− 9.9466	+15.5072	+ 5.5856	− 1.5700	− 2.1323
5	− 9.9878	+15.5402	+ 5.5820	− 0.5442	− 0.7409
6	− 9.6689	+15.0722	+ 5.4291	+ 0.6270	+ 0.8508
7	− 9.2573	+14.4877	+ 5.2492	+ 1.4299	+ 1.9685
8	− 9.3611	+14.6645	+ 5.3207	+ 1.7351	+ 2.3991
9	−10.4173	+16.2394	+ 5.8483	+ 1.6943	+ 2.3074
10	−12.2555	+18.9306	+ 6.7244	+ 1.3762	+ 1.8132
11	−13.7191	+21.0340	+ 7.3897	+ 0.6699	+ 0.8608
	D_0''	$D_{c,1}$	$D_{c,2}$	$D_{s,1}$	$D_{s,2}$
0	+ 1.1897	− 1.3484	− 0.2485	−29.8546	−38.9716
1	+ 3.5868	− 4.2373	− 0.8724	−27.5111	−36.6653
2	+ 4.5580	− 5.5349	− 1.2117	−24.9093	−34.0089
3	+ 4.3152	− 5.3345	− 1.2153	−23.3292	−32.2553
4	+ 3.1531	− 3.9457	− 0.9269	−22.7393	−31.4609
5	+ 1.3233	− 1.7080	− 0.4334	−22.5117	−31.0906
6	− 0.7706	+ 0.8979	+ 0.1689	−22.2069	−30.7822
7	− 2.6659	+ 3.3202	+ 0.7653	−22.0282	−30.7403
8	− 4.0554	+ 5.1025	+ 1.2108	−22.6267	−31.5788
9	− 4.6896	+ 5.8349	+ 1.3576	−24.5324	−33.7503
10	− 4.1057	+ 4.9831	+ 1.1038	−27.4981	−36.8202
11	− 1.9060	+ 2.2816	+ 0.4940	−29.9252	−39.1222

TABLE X.

COEFFICIENTS OF COS 18L AND SIN 18L FOR A, B, C, AND D IN EACH OF 12 SYSTEMS ($L = v - g'$).

System.	$30A_c$	$30A_s$	$30B_c$	$30B_s$	$30C_c$	$30C_s$	$30D_c$	$30D_s$
0	+6.2049	+0.6020	−4.9971	−0.5329	−1.2077	−0.0692	+0.5720	−5.4864
1	+5.1524	+1.6797	−4.2291	−1.4580	−0.9233	−0.2217	+1.5716	−4.6133
2	+4.0908	+2.0174	−3.4091	−1.7277	−0.6816	−0.2896	+1.8673	−3.7002
3	+3.5721	+1.8476	−2.9712	−1.5447	−0.6009	−0.3028	+1.6835	−3.2281
4	+3.5031	+1.3168	−2.8674	−1.0770	−0.6358	−0.2398	+1.1840	−3.1341
5	+3.5273	+0.4896	−2.8577	−0.4074	−0.6697	−0.0821	+0.4448	−3.1352
6	+3.3871	−0.4122	−2.7680	+0.3160	−0.6191	+0.0962	−0.3565	−3.0267
7	+3.1874	−1.1332	−2.6537	+0.9305	−0.5337	+0.2028	−1.0208	−2.8813
8	+3.2562	−1.6267	−2.7313	+1.3868	−0.5250	+0.2401	−1.2991	−2.9577
9	+3.8775	−1.9181	−3.2097	+1.6630	−0.6677	+0.2552	−1.7883	−3.4938
10	+5.0768	−1.7932	−4.1114	+1.5501	−0.9655	+0.2431	−1.6730	−4.5098
11	+6.2050	−0.8671	−4.9641	+0.7360	−1.2409	+0.1311	−0.8012	−5.4650
α_0	+4.2534	+0.0169	−3.4808	−0.0138	−0.7726	−0.0031	+0.0154	−3.8026
α_1	+1.3404	+0.4237	−1.0631	−0.3618	−0.2773	−0.0620	+0.3915	−1.1728
β_1	−0.2170	+1.9499	+0.1657	−1.6577	+0.0512	−0.2923	+1.7968	+0.1855
α_2	+0.5358	+0.0646	−0.3961	−0.0834	−0.1397	+0.0188	+0.0796	−0.4480
β_2	−0.3789	+0.2585	+0.2565	−0.2410	+0.1224	−0.0174	+0.2547	+0.3019
α_3	+0.0682	+0.0800	−0.0512	−0.0602	−0.0170	−0.0197	+0.0698	−0.0569
β_3	−0.0679	+0.0673	+0.0488	−0.0540	+0.0192	−0.0133	+0.0611	+0.0554

The coefficients A_c, A_s, etc., have a separate value for each ot the 12 systems. These special values are developed in a periodic series proceeding according to the sines and cosines of multiples of g', in the form (a) §36 with results shown in the last seven lines above. The final development is then shown below in the form (b).

TABLE XI.

COMPUTATION OF A- AND K-COEFFICIENTS FOR THE HANSENIAN INEQUALITY OF LONG PERIOD.

Arg.	$30A_c$	$30A_s$	$30B_c$	$30B_s$	$30C_c$	$30C_s$	$30D_c$	$30D_s$
18v−18g′	+4.2534	+0.0169	−3.4808	−0.0138	−0.7726	−0.0031	+0.0154	−3.8026
18v−17g′	−0.3048	+0.1034	+0.2973	−0.0981	+0.0075	−0.0054	+0.1030	+0.3120
18v−16g′	+0.1387	−0.1572	−0.0775	+0.0865	−0.0611	+0.0706	−0.1111	−0.0966
18v−15g′	+0.0005	+0.0061	+0.0014	−0.0057	−0.0019	−0.0003	+0.0072	+0.0021
Arg.	$30K_c$	$30K_s$	10^3MK_c	10^3MK_s	10^3MC_c	10^3MC_s	10^3MD_c	10^3MD_s
18v−18g′	+3.8671	+0.0104	+0″.5469	+0″.0015	−0″.1093	−0″.0004	+0″.0022	−0″.5376
18v−17g′	−0.3010	+0.1008	−0.0426	+0.0142	+0.0011	−0.0008	+0.0146	+0.0441
18v−16g′	+0.1081	−0.1218	+0.01528	−0.01723	−0.00864	+0.00998	−0.01571	−0.01366
18v−15g′	−0.0004	+0.0059	−0.00006	+0.00083	−0.00027	−0.00004	+0.00102	+0.00030

§ 37. *Coefficients E and F for Venus.* Some preliminary computations render it doubtful whether the planetary coefficients E and F would lead to sensible inequalities in any case. But, in order to leave no doubt, they are computed for six ot the twelve systems and thirty alternate values of the index for Venus. The separate numerical results are shown in Table XII. The general development will be, so far as it seemed useful to use it, found in Part IV.

TABLE XII.

SPECIAL VALUES OF E AND F FOR THE ACTION OF VENUS ON THE MOON.

	Coefficient E.							Coefficient F.					
i	System 0	System 2	System 4	System 6	System 8	System 10	i	System 0	System 2	System 4	System 6	System 8	System 10
0	+3.528	+6.997	+3.567	−2.569	−6.300	−4.684	0	+0.071	+0.617	+0.231	+0.039	+0.532	+0.345
1	+2.216	+3.683	+1.277	−1.664	−2.541	−1.248	1	−1.154	−1.613	−0.556	+0.836	+1.484	+0.750
2	+0.643	+0.872	+0.160	−0.545	−0.620	−0.150	2	−0.571	+0.701	−0.126	+0.459	+0.574	+0.144
3	+0.218	+0.233	−0.006	−0.198	−0.179	0.000	3	−0.229	−0.226	−0.005	+0.198	+0.192	0.000
4	+0.094	+0.078	−0.024	−0.089	−0.064	+0.019	4	−0.100	−0.077	+0.023	+0.090	+0.069	−0.021
5	+0.049	+0.030	−0.022	−0.046	−0.026	+0.018	5	−0.048	−0.028	+0.020	+0.045	+0.026	−0.019
6	+0.028	+0.012	−0.018	−0.027	−0.024	+0.015	6	−0.025	−0.010	+0.015	+0.024	+0.010	−0.014
7	+0.017	+0.004	−0.014	−0.017	−0.004	+0.013	7	−0.014	+0.003	+0.011	+0.013	+0.003	−0.010
8	+0.011	0.000	−0.012	−0.011	0.000	+0.011	8	−0.007	0.000	+0.007	+0.007	0.000	−0.007
9	+0.007	−0.002	−0.009	−0.007	+0.002	+0.009	9	−0.004	+0.001	+0.005	+0.004	−0.001	−0.005
10	+0.004	−0.003	−0.008	−0.004	+0.003	+0.008	10	−0.002	+0.002	+0.003	+0.002	−0.001	−0.004
11	+0.003	−0.004	−0.007	−0.003	+0.004	+0.006	11	−0.001	+0.001	+0.002	+0.001	−0.001	−0.002
12	+0.001	−0.004	−0.005	−0.001	+0.004	+0.005	12	0.000	+0.001	+0.001	0.000	−0.001	−0.002
13	0.000	−0.005	−0.005	0.000	+0.004	+0.004	13	0.000	+0.001	+0.001	0.000	−0.001	−0.001
14	−0.001	−0.005	−0.004	+0.001	+0.005	+0.004	14	0.000	0.000	0.000	0.000	−0.001	0.000
15	−0.002	−0.005	−0.003	+0.002	+0.005	+0.003	15	0.000	0.000	0.000	0.000	0.000	0.000
16	−0.003	−0.005	−0.002	+0.003	+0.005	+0.002	16	0.000	−0.001	0.000	0.000	0.000	0.000
17	−0.004	−0.005	−0.001	+0.004	+0.005	+0.001	17	−0.001	−0.001	0.000	+0.001	+0.001	0.000
18	−0.005	−0.005	0.000	+0.005	+0.005	0.000	18	−0.001	−0.001	0.000	+0.001	+0.001	0.000
19	−0.006	−0.005	+0.001	+0.006	+0.005	−0.002	19	−0.002	−0.002	0.000	+0.002	+0.002	−0.001
20	−0.008	−0.004	+0.003	+0.007	+0.005	−0.003	20	−0.004	−0.002	+0.001	+0.003	+0.002	−0.002
21	−0.010	−0.004	+0.005	+0.009	+0.004	−0.006	21	−0.006	−0.002	+0.003	+0.005	+0.002	−0.003
22	−0.012	−0.003	+0.009	+0.012	+0.002	−0.010	22	−0.008	−0.001	+0.006	+0.008	+0.002	−0.007
23	−0.016	0.000	+0.014	+0.015	0.000	−0.017	23	−0.012	0.000	+0.011	+0.012	0.000	−0.012
24	−0.021	+0.005	+0.024	+0.020	−0.007	−0.029	24	−0.019	+0.005	+0.022	+0.017	−0.006	−0.025
25	−0.028	+0.018	+0.043	+0.027	−0.023	−0.053	25	−0.028	+0.018	+0.043	+0.026	−0.021	−0.050
26	−0.039	+0.050	+0.084	+0.037	−0.065	−0.108	26	−0.042	+0.055	+0.090	+0.038	−0.064	−0.109
27	−0.050	+0.152	+0.194	+0.045	−0.208	−0.264	27	−0.053	+0.167	+0.205	+0.044	−0.198	−0.260
28	0.000	+0.569	+0.561	−0.017	−0.799	−0.819	28	0.000	+0.543	+0.507	−0.014	−0.623	−0.672
29	+0.730	+2.564	+1.885	−0.681	−3.325	−2.919	29	+0.406	+1.557	+1.061	−0.325	−1.402	−1.331

B. ACTION OF MARS.

§ 38. For Mars the coefficients A, B, C, and D were developed much in the same way as for Venus. But, owing to the supposed absence of terms having a high multiple of the mean longitude of Mars, it was considered sufficient to divide the mean orbit of Mars into 24 parts for the special computations of the A-coefficients. The adopted number of systems was 12, as in the case of Venus.

The following statements, with the diagram, will make clear the method of carrying out the computation. In system 0 the Earth remains at rest at its perihelion and is, therefore, in longitude approximately $\pi_0' = 99^\circ.5$. Mars starting from this same mean longitude, π_0', takes the twenty-four consecutive mean longitudes π_0', $\pi_0' + 15^\circ$, $\pi_0' + 30^\circ$, etc., to $\pi^{0\prime} + 345^\circ$. These twenty-four positions are designated by the twenty-four indices 0, 1, 2, 3, . . . 23.

In system 1 the Earth is in mean anomaly 30°. Then, as before, Mars takes the successive mean longitudes $\pi_0' + 30^\circ$, $\pi_0' + 45^\circ$, . . . up to $\pi_0' + 15^\circ$.

The same plan is carried through; the constant mean anomaly of the Earth in the ith system being $i \times 30^\circ$, while Mars, starting with the same mean longitude, goes through its twenty-four consecutive mean positions, the indices which express the mean longitude of Mars always starting with the value 0 when Mars is in mean conjunction with the Earth.

As in the case of Venus, the elements were taken with their values for 1800, in order to correspond to the mean of the period during which the longitude of the Moon has been observed. The numbers and data for computing the longitude of Mars are, then, as follows:

π_0'; long. of ⊕'s perihelion for 1800;	$99^\circ 30' \ 7''.6$
π_4; " " Mars' " " " ;	332 22 42 .9
$\pi_0' - \pi_4$; initial mean anom. of Mars for 1800;	127 7 24 .7

Initial mean anomaly of Mars in system j

$$g_4 = 127^\circ 7' 24''.7 + 30^\circ \times j$$

For system j and index i

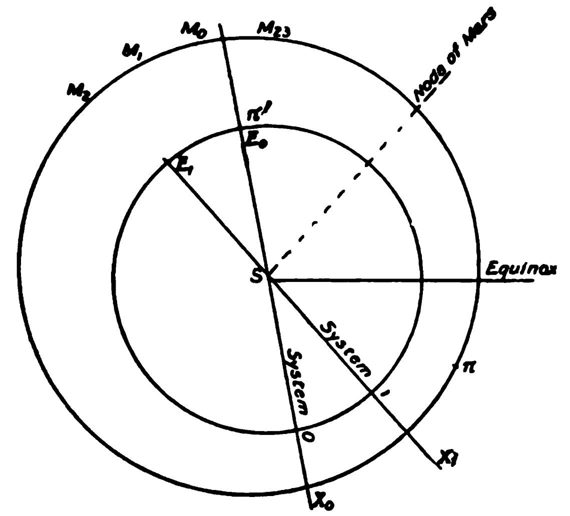
$$g_4 = 127^\circ 7' 24''.7 + 30^\circ \times j + 15^\circ \times i$$

From the numbers found in *Tables of Mars*, page 397, it is found that to this initial mean anomaly corresponds

$$\text{Fund. Arg. } N = 243^d.0948$$

and that the increment of N for 15° of mean anomaly is

$$\Delta N = 28^d.62482$$



Arrangement of Coördinate Axes in Systems 0, 1, etc., for Mars.

For the numbers arising from the inclination of the orbit of Mars we have:

Long. of node, 1800; . $\theta = 48^\circ \ 0' \ 52''.5$

Node from ⊕'s perihelion; $308^\circ 30' 44''.9$

Inclination, 1800; . . . . $I = 1^\circ 51' \ 3''.6$

The results of the main steps in the computation of the coördinates of Mars are shown in the following table. The first column corresponds to the indices of system 0. In the *j*th system they are diminished by 2*j*.

The second column shows the value of *N* actually used in entering the tables. The discrepancy of two units in the fourth place results from using two computations of *N*. Column *f* gives the mean anomaly as taken from the tables, reduced by the secular variation to 1800. Column *u* is formed by adding to *f* the distance from the node to the perihelion of Mars and applying the reduction to the ecliptic. This reduction was applied in order to use for *x* and *y* simple formulæ for the ecliptic longitude. Actually, through a misapprehension, the rectangular coördinates were computed on the supposition that *u* was counted along the orbit, as in the case of Venus. There is therefore an error in the last figures of the coördinates, the amount of which can readily be determined, but which has been deemed too small to need correction for the present problem.

TABLE XIII.

COMPUTATION OF HELIOCENTRIC COÖRDINATES OF MARS.

i	N	*f*			*u*			log. *r*	*x*	*y*	*z*
		°	′	″							
0	243.0946	135	3	17	59	24	20	0.208 749	−1.601 16	−0.222 32	+0.044 97
1	271.7194	148	7	1	72	28	19	0.214 918	−1.530 86	−0.586 91	+0.050 53
2	300.3442	160	52	11	85	13	52	0.219 143	−1.376 61	−0.919 51	+0.053 32
3	328.9690	173	26	7	97	48	12	0.221 310	−1.149 04	−1.203 24	+0.053 27
4	357.5939	185	56	13	110	18	38	0.221 364	−0.861 21	−1.423 86	+0.050 43
5	386.2187	198	29	47	122	52	26	0.219 303	−0.528 30	−1.569 81	+0.044 05
6	414.8435	211	14	12	135	36	56	0.215 179	−0.167 45	−1.632 28	+0.037 08
7	443.4683	224	16	54	148	39	31	0.209 102	+0.202 38	−1.605 53	+0.027 10
8	472.0932	237	45	02	162	7	24	0.201 264	+0.566 38	−1.487 37	+0.015 70
9	500.7180	251	45	55	176	7	52	0.191 962	+0.884 48	−1.270 95	+0.003 30
10	529.3428	266	25	53	190	47	23	0.181 634	+1.151 73	−0.990 70	−0.009 10
11	557.9676	281	49	50	206	10	58	0.170 884	+1.339 04	−0.633 72	−0.021 13
12	586.5924	298	0	12	222	21	9	0.160 502	+1.428 36	−0.230 00	−0.031 40
13	615.2172	314	55	23	239	16	26	0.151 419	+1.403 60	+0.191 54	−0.039 35
14	643.8421	332	28	46	256	50	12	0.144 609	+1.266 23	+0.569 80	−0.043 88
15	672.4669	350	28	12	274	50	11	0.140 910	+1.005 34	+0.919 00	−0.044 52
16	14.0961	8	37	13	292	59	42	0.140 817	+0.659 23	+1.215 07	−0.041 12
17	42.7209	26	37	36	311	00	19	0.144 343	+0.253 23	+1.370 05	−0.033 99
18	71.3457	44	12	35	328	35	12	0.151 014	−0.175 28	+1.404 75	−0.023 81
19	99.9705	61	9	53	345	32	8	0.160 006	−0.589 17	+1.319 88	−0.011 66
20	128.5954	77	22	34	1	44	20	0.170 346	−0.956 46	+1.129 70	+0.001 48
21	157 2202	92	48	50	17	10	10	0.181 098	−1.253 06	+0.855 04	+0.014 47
22	185.8450	107	30	54	31	51	56	0.191 465	−1.463 47	+0.522 14	+0.026 50
23	214.4698	121	33	36	45	54	32	0.200 829	−1.570 94	+0.154 73	+0.036 81

TABLE XIV.

G-COÖRDINATES OF MARS REDUCED TO THE DIFFERENT SYSTEMS.

	System 0.		System 1.		System 2.		System 3.	
	X	*Y*	*X*	*Y*	*X*	*Y*	*X*	*Y*
0	−0.617 95	−0.222 32	−0.666 54	−0.091 17	−0.672 31	+0.063 05	−0.632 56	+0.201 03
1	−0.547 65	−0.586 91	−0.611 33	−0.450 67	−0.632 25	−0.298 23	−0.605 81	−0.168 80
2	−0.393 40	−0.919 51	−0.472 37	−0.785 65	−0.505 93	−0.641 98	−0.487 65	−0.526 80
3	−0.165 83	−1.203 24	−0.257 03	−1.078 49	−0.297 85	−0.948 87	−0.280 23	−0.850 90
4	+0.122 00	−1.423 86	+0.024 24	−1.313 03	−0.016 52	−1.199 84	+0.008 96	−1.118 15
5	+0.454 91	−1.569 81	+0.357 89	−1.474 77	+0.325 16	−1.376 81	+0.366 00	−1.306 06
6	+0.815 76	−1.632 28	+0.727 02	−1.351 44	+0.709 23	−1.463 66	+0.769 66	−1.394 78
7	+1.185 59	−1.605 53	+1.111 41	−1.533 86	+1.112 39	−1.447 88	+1.191 26	−1.370 02
8	+1.543 59	−1.487 37	+1.487 44	−1.417 03	+1.506 33	−1.322 88	+1.596 58	−1.226 65
9	+1.867 69	−1.279 95	+1.828 70	−1.201 79	+1.859 07	−1.090 63	+1.948 81	−0.971 76
10	+2.134 94	−0.990 76	+2.107 36	−0.896 57	+2.138 39	−0.763 82	+2.214 79	−0.625 65
11	+2.322 85	−0.633 72	+2.296 71	−0.519 07	+2.316 00	−0.366 96	+2.370 37	−0.219 65
12	+2.411 57	−0.230 06	+2.375 22	−0.096 37	+2.373 28	+0.065 77	+2.404 47	+0.208 86
13	+2.386 81	+0.191 54	+2.330 58	+0.336 12	+2.305 02	+0.495 16	+2.319 60	+0.622 75
14	+2.243 44	+0.596 86	+2.163 83	+0.739 52	+2.120 30	+0.883 32	+2.129 51	+0.990 04
15	+1.988 55	+0.949 09	+1.890 02	+1.077 26	+1.839 87	+1.199 33	+1.855 35	+1.286 64
16	+1.642 44	+1.215 07	+1.535 96	+1.321 04	+1.491 59	+1.422 36	+1.521 85	+1.497 05
17	+1.236 44	+1.370 65	+1.135 09	+1.454 49	+1.105 86	+1.542 15	+1.154 45	+1.613 52
18	+0.807 93	+1.404 75	+0.721 96	+1.473 51	+0.711 84	+1.557 61	+0.777 40	+1.634 74
19	+0.394 04	+1.319 88	+0.328 01	+1.384 38	+0.335 42	+1.474 78	+0.412 81	+1.564 44
20	+0.026 75	+1.129 79	−0.020 95	+1.200 76	−0.001 73	+1.304 64	+0.080 21	+1.410 19
21	−0.269 85	+0.855 63	−0.305 52	+0.940 82	−0.282 32	+1.061 47	−0.203 52	+1.182 62
22	−0.480 26	+0.522 13	−0.512 42	+0.624 89	−0.493 23	+0.761 58	−0.424 14	+0.894 79
23	−0.596 73	+0.154 73	−0.633 83	+0.274 00	−0.625 17	+0.422 63	−0.570 09	+0.561 88

	System 4.		System 5.		System 6.		System 7.	
0	−0.560 10	+0.287 39	−0.478 34	+0.298 89	−0.411 57	+0.230 06	−0.375 36	+0.096 49
1	−0.542 52	−0.097 00	−0.462 56	−0.104 27	−0.386 81	−0.191 54	−0.330 72	−0.336 00
2	−0.425 69	−0.473 03	−0.337 56	−0.498 21	−0.243 44	−0.596 86	−0.163 97	−0.739 40
3	−0.210 45	−0.814 29	−0.105 31	−0.850 95	+0.011 45	−0.949 09	+0.109 84	−1.077 14
4	+0.094 77	−1.092 95	+0.221 50	−1.130 27	+0.357 56	−1.215 07	+0.463 90	−1.320 92
5	+0.472 27	−1.282 30	+0.618 36	−1.307 88	+0.763 56	−1.370 65	+0.864 77	−1.454 37
6	+0.894 97	−1.360 81	+1.051 09	−1.365 16	+1.192 07	−1.404 75	+1.277 90	−1.473 39
7	+1.327 46	−1.316 17	+1.480 48	−1.296 90	+1.605 96	−1.319 88	+1.671 85	−1.384 26
8	+1.730 86	−1.149 42	+1.868 64	−1.112 18	+1.973 25	−1.129 79	+2.020 81	−1.200 64
9	+2.068 60	−0.875 61	+2.184 65	−0.831 75	+2.269 85	−0.855 63	+2.305 38	−0.940 70
10	+2.312 38	−0.521 55	+2.407 68	−0.483 47	+2.480 26	−0.522 13	+2.512 28	−0.624 77
11	+2.445 83	−0.120 68	+2.527 47	−0.097 74	+2.596 73	−0.154 73	+2.633 69	−0.273 88
12	+2.464 85	+0.292 45	+2.542 93	+0.296 28	+2.617 95	+0.222 32	+2.666 40	+0.091 29
13	+2.375 72	+0.686 40	+2.460 10	+0.672 70	+2.547 65	+0.586 91	+2.611 19	+0.450 79
14	+2.192 10	+1.035 36	+2.289 96	+1.009 85	+2.393 40	+0.919 51	+2.472 23	+0.785 77
15	+1.932 16	+1.319 93	+2.046 79	+1.290 44	+2.165 83	+1.203 24	+2.256 89	+1.078 61
16	+1.616 23	+1.526 83	+1.746 90	+1.501 35	+1.878 00	+1.423 86	+1.975 62	+1.313 15
17	+1.265 34	+1.648 24	+1.407 95	+1.633 29	+1.545 09	+1.569 81	+1.641 97	+1.474 89
18	+0.900 17	+1.680 95	+1.048 37	+1.680 43	+1.184 24	+1.632 28	+1.272 84	+1.551 56
19	+0.540 67	+1.625 74	+0.687 09	+1.640 37	+0.814 41	+1.605 53	+0.888 45	+1.533 98
20	+0.205 69	+1.486 78	+0.343 34	+1.514 05	+0.456 41	+1.487 37	+0.512 42	+1.417 15
21	−0.087 15	+1.271 44	+0.036 45	+1.305 97	+0.132 31	+1.279 95	+0.171 16	+1.201 91
22	−0.321 69	+0.990 17	−0.214 52	+1.024 64	−0.134 04	+0.990 76	−0.107 50	+0.896 69
23	−0.483 43	+0.656 52	−0.391 49	+0.682 96	−0.322 85	+0.633 72	−0.296 85	+0.519 19

TABLE XIV.—*Concluded.*

G-COÖRDINATES OF MARS REDUCED TO THE DIFFERENT SYSTEMS.

	System 8.		System 9.		System 10.		System 11.	
	X	*Y*	*X*	*Y*	*X*	*Y*	*X*	*Y*
0	−0.373 70	−0.065 64	−0.405 03	−0.208 86	−0.465 27	−0.292 58	−0.543 07	−0.296 40
1	−0.305 44	−0.495 03	−0.320 16	−0.622 75	−0.376 14	−0.686 53	−0.460 24	−0.672 82
2	−0.120 72	−0.883 19	−0.130 07	−0.990 04	−0.192 52	−1.035 49	−0.290 10	−1.009 97
3	+0.159 71	−1.199 20	+0.144 09	−1.286 64	+0.067 42	−1.320 06	−0.046 93	−1.290 56
4	+0.507 99	−1.422 23	+0.477 59	−1.497 05	+0.383 35	−1.526 96	+0.252 96	−1.501 47
5	+0.893 72	−1.542 02	+0.844 99	−1.613 52	+0.734 24	−1.648 37	+0.591 91	−1.633 41
6	+1.287 74	−1.557 48	+1.222 04	−1.634 74	+1.099 41	−1.681 08	+0.951 49	−1.680 55
7	+1.664 16	−1.474 65	+1.586 63	−1.564 44	+1.458 91	−1.625 87	+1.312 77	−1.640 49
8	+2.001 31	−1.304 51	+1.919 23	−1.410 19	+1.793 89	−1.486 91	+1.656 52	−1.514 17
9	+2.281 90	−1.061 34	+2.202 96	−1.182 62	+2.086 73	−1.271 57	+1.963 41	−1.306 09
10	+2.492 81	−0.761 45	+2.423 58	−0.894 79	+2.321 27	−0.990 30	+2.214 38	−1.024 76
11	+2.624 75	−0.422 50	+2.569 53	−0.561 88	+2.483 01	−0.656 65	+2.391 35	−0.683 08
12	+2.671 89	−0.062 92	+2.632 00	−0.201 03	+2.559 68	−0.287 52	+2.478 20	−0.299 01
13	+2.631 83	+0.298 36	+2.605 25	+0.168 80	+2.542 10	+0.096 87	+2.462 42	+0.104 15
14	+2.505 51	+0.642 11	+2.487 09	+0.526 80	+2.425 27	+0.472 90	+2.337 42	+0.498 09
15	+2.297 43	+0.949 00	+2.279 67	+0.850 90	+2.210 03	+0.814 16	+2.105 17	+0.850 83
16	+2.016 10	+1.199 97	+1.990 48	+1.118 15	+1.904 81	+1.092 82	+1.778 36	+1.130 15
17	+1.674 42	+1.376 94	+1.633 44	+1.306 06	+1.527 31	+1.282 17	+1.381 50	+1.307 76
18	+1.290 35	+1.463 79	+1.229 78	+1.394 78	+1.104 61	+1.360 68	+0.948 77	+1.365 04
19	+0.887 19	+1.448 01	+0.808 18	+1.370 02	+0.672 12	+1.316 04	+0.519 38	+1.296 78
20	+0.493 25	+1.323 01	+0.402 86	+1.226 65	+0.268 72	+1.149 29	+0.131 22	+1.112 06
21	+0.140 51	+1.090 76	+0.050 63	+0.971 76	−0.069 02	+0.875 48	−0.184 79	+0.831 63
22	−0.138 81	+0.763 95	−0.215 35	+0.625 65	−0.312 80	+0.521 42	−0.407 82	+0.483 35
23	−0.316 42	+0.367 09	−0.370 93	+0.219 65	−0.446 25	+0.120 55	−0.527 61	+0.097 62

TABLE XV.

SPECIAL VALUES OF A, B, C, AND D FOR MARS.

System 0.					System 1.				
i	*A*	*B*	*C*	*D*	*i*	*A*	*B*	*C*	*D*
0	+1.920 41	−0.768 57	−1.151 95	+1.111 16	0	+2.088 60	−1.024 97	−1.064 03	+0.433 99
1	+0.250 33	+0.382 65	−0.633 03	+0.954 82	1	+0.705 52	+0.038 62	−0.744 19	+1.076 90
2	−0.178 22	+0.507 19	−0.328 97	+0.358 95	2	−0.088 66	+0.515 53	−0.426 83	+0.568 93
3	−0.175 15	+0.359 60	−0.184 43	+0.075 13	3	−0.204 72	+0.447 51	−0.242 79	+0.164 80
4	−0.111 50	+0.225 06	−0.113 56	−0.029 05	4	−0.146 86	+0.293 51	−0.146 66	−0.008 13
5	−0.058 53	+0.134 50	−0.076 06	−0.061 07	5	−0.079 42	+0.174 65	−0.095 23	−0.065 51
6	−0.021 97	+0.076 72	−0.054 76	−0.065 74	6	−0.030 47	+0.096 73	−0.066 25	−0.076 39
7	+0.002 45	+0.039 44	−0.041 89	−0.060 08	7	+0.001 61	+0.047 44	−0.049 04	−0.069 91
8	+0.018 80	+0.015 03	−0.033 83	−0.050 72	8	+0.022 02	+0.016 43	−0.038 44	−0.057 60
9	+0.029 90	−0.001 19	−0.028 72	−0.040 17	9	+0.034 83	−0.003 03	−0.031 80	−0.043 79
10	+0.037 54	−0.011 98	−0.025 57	−0.029 29	10	+0.042 73	−0.014 99	−0.027 72	−0.029 98
11	+0.042 79	−0.018 92	−0.023 87	−0.018 19	11	+0.047 31	−0.021 81	−0.025 51	−0.016 46
12	+0.046 23	−0.022 81	−0.023 43	−0.006 65	12	+0.049 46	−0.024 68	−0.024 77	−0.003 01
13	+0.048 05	−0.023 80	−0.024 25	+0.005 80	13	+0.049 46	−0.023 96	−0.025 49	+0.010 81
14	+0.047 95	−0.021 35	−0.026 60	+0.019 84	14	+0.046 97	−0.019 12	−0.027 83	+0.025 58
15	+0.044 92	−0.013 81	−0.031 10	+0.036 30	15	+0.040 91	−0.008 56	−0.032 35	+0.041 77
16	+0.036 65	+0.002 37	−0.039 01	+0.056 01	16	+0.029 03	+0.011 04	−0.040 06	+0.059 44
17	+0.018 30	+0.034 61	−0.052 92	+0.079 00	17	+0.007 19	+0.045 88	−0.053 06	+0.077 22
18	−0.019 93	+0.098 16	−0.078 25	+0.101 49	18	−0.031 63	+0.107 09	−0.075 45	+0.089 44
19	−0.096 22	+0.223 72	−0.127 50	+0.104 86	19	−0.097 28	+0.212 99	−0.115 69	+0.077 88
20	−0.230 56	+0.461 52	−0.230 95	+0.016 40	20	−0.192 14	+0.384 18	−0.192 04	−0.010 06
21	−0.336 21	+0.797 25	−0.461 05	−0.396 96	21	−0.245 44	+0.587 62	−0.342 21	−0.302 42
22	+0.347 58	+0.580 43	−0.927 87	−1.390 90	22	+0.127 35	+0.495 62	−0.622 90	−0.922 00
23	+2.549 00	−1.148 74	−1.400 16	−1.027 90	23	+1.520 82	−0.532 46	−0.988 31	−1.091 63

TABLE XV.—*Continued.*

SPECIAL VALUES OF A, B, C, AND D FOR MARS.

System 2.					System 3.				
i	*A*	*B*	*C*	*D*	*i*	*A*	*B*	*C*	*D*
0	+2.101 41	−1.045 67	−1.055 73	−0.297 76	0	+1.947 47	−0.823 38	−1.124 08	−0.979 52
1	+1.400 17	−0.442 44	−0.957 75	+1.117 88	1	+2.376 80	−1.048 03	−1.328 82	+1.034 57
2	+0.089 45	+0.515 34	−0.604 76	+0.885 66	2	+0.345 55	+0.553 68	−0.899 15	+1.346 03
3	−0.247 49	+0.585 27	−0.337 75	+0.289 97	3	−0.327 52	+0.791 16	−0.463 59	+0.413 24
4	−0.192 78	+0.385 57	−0.192 79	+0.007 96	4	−0.238 32	+0.476 68	−0.238 32	−0.005 73
5	−0.099 08	+0.216 83	−0.117 74	−0.079 02	5	−0.104 37	+0.237 83	−0.133 43	−0.104 06
6	−0.033 27	+0.110 73	−0.077 45	−0.091 19	6	−0.024 71	+0.107 04	−0.082 30	−0.104 54
7	+0.006 21	+0.048 52	−0.054 73	−0.079 34	7	+0.016 21	+0.039 39	−0.055 58	−0.082 65
8	+0.028 67	+0.012 65	−0.041 33	−0.061 50	8	+0.036 14	+0.004 61	−0.040 75	−0.059 12
9	+0.040 97	−0.007 72	−0.033 24	−0.043 55	9	+0.045 21	−0.013 00	−0.032 22	−0.038 63
10	+0.047 22	−0.018 80	−0.028 43	−0.027 03	10	+0.048 58	−0.021 27	−0.027 30	−0.021 44
11	+0.049 75	−0.023 94	−0.025 81	−0.011 98	11	+0.048 76	−0.024 07	−0.024 68	−0.006 81
12	+0.049 71	−0.024 83	−0.024 87	+0.002 07	12	+0.046 87	−0.023 18	−0.023 70	+0.006 13
13	+0.047 48	−0.022 06	−0.025 41	+0.015 66	13	+0.043 26	−0.019 21	−0.024 06	+0.018 07
14	+0.042 80	−0.015 30	−0.027 49	+0.029 29	14	+0.037 75	−0.012 02	−0.025 74	+0.029 52
15	+0.034 78	−0.003 32	−0.031 46	+0.043 18	15	+0.029 70	−0.000 75	−0.028 96	+0.040 68
16	+0.021 75	+0.016 33	−0.038 07	+0.057 04	16	+0.017 96	+0.016 27	−0.034 24	+0.051 37
17	+0.000 92	+0.047 85	−0.048 76	+0.069 29	17	+0.000 66	+0.041 96	−0.042 62	+0.060 55
18	−0.031 96	+0.098 26	−0.066 29	+0.075 22	18	−0.025 10	+0.081 14	−0.056 05	+0.065 29
19	−0.082 07	+0.178 16	−0.096 09	+0.062 41	19	−0.063 25	+0.141 60	+0.078 35	+0.058 10
20	−0.149 84	+0.299 14	−0.149 31	−0.000 60	20	−0.116 89	+0.234 43	−0.117 53	+0.020 05
21	−0.201 17	+0.450 32	−0.249 17	−0.186 47	21	−0.175 77	+0.366 96	−0.191 20	−0.096 25
22	−0.051 76	+0.491 04	−0.439 33	−0.605 51	22	−0.154 23	+0.493 56	−0.339 31	−0.396 04
23	+0.807 50	−0.049 04	−0.758 48	−1.066 38	23	+0.334 45	+0.306 39	−0.640 81	−0.967 23
System 4.					System 5.				
0	+1.832 10	−0.500 77	−1.331 96	−1.624 80	0	+2.148 15	−0.293 03	−1.855 19	−2.502 41
1	+3.797 02	−1.806 10	−1.990 90	+1.034 88	1	+5.764 33	−2.665 97	−3.098 75	+2.002 10
2	+0.442 09	+0.849 47	−1.292 02	+1.927 87	2	−0.089 37	+1.600 16	−1.510 72	+2.116 20
3	−0.454 66	+1.013 21	−0.558 66	+0.406 52	3	−0.503 27	+1.027 00	−0.523 75	+0.192 32
4	−0.246 47	+0.497 94	−0.251 49	−0.065 04	4	−0.193 56	+0.410 28	−0.216 73	−0.123 06
5	−0.083 76	+0.213 91	−0.130 15	−0.126 83	5	−0.049 74	+0.159 37	−0.109 63	−0.127 33
6	−0.007 31	+0.084 19	−0.076 89	−0.106 05	6	+0.007 54	+0.057 46	−0.065 01	−0.094 38
7	+0.026 10	+0.024 79	−0.050 89	−0.076 42	7	+0.030 46	+0.013 21	−0.043 66	−0.064 96
8	+0.040 16	−0.003 05	−0.037 10	−0.051 33	8	+0.039 37	−0.006 98	−0.032 40	−0.042 72
9	+0.045 38	−0.016 00	−0.029 38	−0.031 65	9	+0.042 28	−0.016 18	−0.026 09	−0.026 03
10	+0.046 40	−0.021 39	−0.025 01	−0.016 11	10	+0.042 40	−0.019 89	−0.022 51	−0.013 03
11	+0.045 24	−0.022 53	−0.022 70	−0.003 35	11	+0.041 09	−0.020 51	−0.020 60	−0.002 39
12	+0.042 68	−0.020 89	−0.021 80	+0.007 65	12	+0.038 92	−0.019 06	−0.019 85	+0.006 85
13	+0.038 98	−0.016 95	−0.022 04	+0.017 63	13	+0.035 97	−0.015 89	−0.020 07	+0.015 33
14	+0.033 97	−0.010 59	−0.023 38	+0.027 09	14	+0.032 11	−0.010 87	−0.021 23	+0.023 53
15	+0.027 18	−0.001 19	−0.025 99	+0.035 51	15	+0.026 95	−0.003 45	−0.023 48	+0.031 81
16	+0.017 72	+0.012 55	−0.030 27	+0.045 37	16	+0.019 75	+0.007 46	−0.027 21	+0.040 39
17	+0.004 14	+0.032 00	−0.037 05	+0.053 75	17	+0.009 24	+0.023 92	−0.033 15	+0.049 24
18	−0.015 94	+0.063 85	−0.047 91	+0.059 91	18	−0.006 86	+0.049 63	−0.042 78	+0.057 70
19	−0.046 43	+0.112 41	−0.065 99	+0.059 39	19	−0.032 71	+0.091 80	−0.059 09	+0.063 25
20	−0.092 87	+0.190 96	−0.098 08	+0.040 04	20	−0.075 96	+0.164 81	−0.088 86	+0.057 56
21	−0.158 48	+0.318 61	−0.160 13	−0.032 86	21	−0.149 03	+0.298 19	−0.149 19	+0.012 49
22	−0.357 45	+1.200 62	−0.843 21	−1.004 68	22	−0.253 88	+0.544 16	−0.290 26	−0.174 74
23	+0.032 96	+0.579 17	−0.612 03	−0.878 68	23	−0.176 33	+0.859 51	−0.683 21	−0.884 34

TABLE XV.—*Continued.*

SPECIAL VALUES OF A, B, C, AND D FOR MARS.

System 6.					System 7.				
i	*A*	*B*	*C*	*D*	*i*	*A*	*B*	*C*	*D*
0	+4.029 84	−0.912 73	−3.116 74	−4.018 27	0	+9.990 63	−4.586 21	−5.404 24	−4.012 20
1	+5.687 73	−1.695 28	−3.992 42	+4.843 56	1	+1.453 90	+1.601 76	−3.055 56	+4.666 10
2	−0.709 53	+1.928 11	−1.218 70	+1.290 50	2	−0.656 79	+1.413 86	−0.757 15	+0.482 93
3	−0.388 33	+0.774 20	−0.385 94	−0.014 03	3	−0.254 14	+0.515 57	−0.201 46	−0.079 32
4	−0.124 66	+0.287 92	−0.163 25	−0.132 92	4	−0.081 45	+0.202 75	−0.121 32	−0.113 85
5	−0.024 99	+0.111 12	−0.086 12	−0.109 95	5	−0.014 79	+0.083 51	−0.068 73	−0.090 63
6	+0.013 63	+0.039 63	−0.053 26	−0.078 85	6	+0.012 94	+0.032 00	−0.044 93	−0.066 72
7	+0.029 34	+0.007 77	−0.037 10	−0.054 60	7	+0.025 42	+0.007 18	−0.032 59	−0.048 03
8	+0.035 71	−0.007 35	−0.028 35	−0.036 68	8	+0.031 23	−0.005 58	−0.025 65	−0.033 80
9	+0.037 98	−0.014 63	−0.023 35	−0.023 12	9	+0.033 91	−0.012 35	−0.021 58	−0.022 65
10	+0.038 32	−0.017 86	−0.020 46	−0.012 38	10	+0.035 04	−0.015 85	−0.019 18	−0.013 49
11	+0.037 65	−0.018 73	−0.018 92	−0.003 37	11	+0.035 30	−0.017 36	−0.017 92	−0.005 54
12	+0.036 33	−0.017 98	−0.018 35	+0.004 65	12	+0.035 00	−0.017 48	−0.017 52	+0.001 80
13	+0.034 45	−0.015 82	−0.018 62	+0.012 23	13	+0.034 24	−0.016 35	−0.017 88	+0.009 00
14	+0.031 88	−0.012 14	−0.019 73	+0.019 84	14	+0.032 90	−0.013 83	−0.019 06	+0.016 52
15	+0.028 28	−0.006 41	−0.021 87	+0.027 88	15	+0.030 68	−0.009 42	−0.021 27	+0.024 84
16	+0.023 00	+0.002 40	−0.025 42	+0.036 73	16	+0.026 96	−0.002 02	−0.024 94	+0.034 51
17	+0.014 84	+0.016 27	−0.031 11	+0.046 70	17	+0.020 46	+0.010 52	−0.030 99	+0.046 22
18	+0.001 39	+0.039 19	−0.040 59	+0.057 91	18	+0.008 52	+0.032 70	−0.041 23	+0.060 66
19	−0.022 05	+0.079 14	−0.057 07	+0.069 11	19	+0.008 53	+0.032 71	−0.041 24	+0.060 67
20	−0.065 65	+0.154 13	−0.088 47	+0.074 45	20	−0.063 61	+0.160 98	−0.097 38	+0.093 42
21	−0.154 98	+0.315 09	−0.160 14	+0.049 51	21	−0.175 09	+0.361 07	−0.186 02	+0.077 93
22	−0.315 17	+0.648 46	−0.333 30	−0.133 72	22	−0.432 50	+0.882 46	−0.450 02	−0.159 94
23	−0.353 82	+1.276 69	−0.922 91	−1.121 87	23	−0.408 83	+1.937 10	−1.528 17	−1.992 73

System 8.					System 9.				
0	+11.253 07	−5.464 71	−5.788 03	+3.030 00	0	+4.782 03	−1.303 27	−3.478 85	+4.274 64
1	− 0.295 87	+1.963 85	−1.667 47	+2.251 42	1	−0.362 00	+1.331 70	−0.969 66	+1.183 54
2	− 0.444 25	+0.913 37	−0.469 09	+0.189 10	2	+0.317 73	+0.652 46	−0.334 77	+0.129 70
3	− 0.178 40	+0.366 62	−0.188 19	−0.073 90	3	−0.147 87	+0.301 39	−0.153 52	−0.050 95
4	− 0.063 93	+0.160 71	−0.096 78	−0.091 97	4	−0.062 08	+0.147 89	−0.085 80	−0.074 57
5	− 0.014 47	+0.073 33	−0.058 87	−0.076 63	5	−0.019 55	+0.074 61	−0.055 06	−0.067 95
6	+ 0.008 79	+0.031 56	−0.040 36	−0.059 48	6	+0.002 94	+0.036 19	−0.039 12	−0.056 34
7	+ 0.020 61	+0.009 67	−0.030 29	−0.045 12	7	+0.015 67	+0.014 40	−0.030 06	−0.045 14
8	+ 0.027 00	−0.002 58	−0.024 40	−0.033 52	8	+0.023 35	+0.001 26	−0.024 62	−0.035 28
9	+ 0.030 62	−0.009 76	−0.020 87	−0.023 97	9	+0.028 30	−0.007 01	−0.021 28	−0.026 63
10	+ 0.032 79	−0.014 00	−0.018 79	−0.015 76	10	+0.031 67	−0.012 37	−0.019 30	−0.018 82
11	+ 0.034 09	−0.016 39	−0.017 71	−0.008 34	11	+0.034 10	−0.015 80	−0.018 29	−0.011 46
12	+ 0.034 85	−0.017 42	−0.017 43	−0.001 23	12	+0.035 91	−0.017 80	−0.018 11	−0.004 13
13	+ 0.035 17	−0.017 25	−0.017 91	+0.006 02	13	+0.037 22	−0.018 50	−0.018 72	+0.003 63
14	+ 0.034 95	−0.015 70	−0.019 25	+0.013 89	14	+0.037 96	−0.017 67	−0.020 28	+0.012 31
15	+ 0.033 90	−0.012 21	−0.021 69	+0.022 97	15	+0.037 78	−0.014 65	−0.023 14	+0.022 74
16	+ 0.031 36	−0.005 56	−0.025 81	+0.034 03	16	+0.035 87	−0.007 85	−0.028 01	+0.035 88
17	+ 0.026 84	+0.006 88	−0.032 72	+0.048 15	17	+0.030 24	+0.006 33	−0.036 58	+0.053 56
18	+ 0.013 99	+0.030 88	−0.044 86	+0.066 76	18	+0.016 16	+0.035 63	−0.051 78	+0.077 10
19	− 0.012 33	+0.080 37	−0.068 02	+0.090 94	19	−0.018 70	+0.101 31	−0.082 60	+0.108 59
20	− 0.075 03	+0.193 18	−0.118 15	+0.116 14	20	−0.109 51	+0.263 59	−0.154 08	+0.137 35
21	− 0.237 87	+0.486 97	−0.249 14	+0.094 95	21	−0.357 72	+0.716 16	−0.358 39	+0.056 10
22	− 0.640 95	+1.342 80	−0.701 91	−0.372 76	22	−0.781 89	+1.912 71	−1.130 82	−1.052 12
23	+ 0.775 27	+2.043 47	−2.818 93	−4.253 90	23	+4.977 43	−0.931 51	−4.045 98	−5.388 50

TABLE XV.—*Concluded.*

SPECIAL VALUES OF A, B, C, AND D FOR MARS.

System 10.					System 11.				
i	*A*	*B*	*C*	*D*	*i*	*A*	*B*	*C*	*D*
0	+ 2.308 84	−0.300 86	−2.007 82	+2.714 50	0	+1.835 03	−0.438 93	−1.395 96	+1.767 63
1	− 0.213 64	+0.907 37	−0.693 77	+0.877 62	1	−0.028 01	+0.637 67	−0.609 70	+0.855 84
2	− 0.256 48	+0.541 01	−0.284 49	+0.153 58	2	−0.221 09	+0.506 01	−0.284 90	+0.227 63
3	− 0.143 04	+0.286 86	−0.143 82	−0.022 01	3	−0.153 79	+0.307 48	−0.153 69	+0.106 80
4	− 0.069 06	+0.152 61	−0.083 56	−0.058 96	4	−0.086 46	+0.180 40	−0.093 91	−0.046 27
5	− 0.028 52	+0.085 09	−0.056 51	−0.063 13	5	−0.041 39	+0.104 70	−0.063 30	−0.060 95
6	− 0.004 18	+0.045 19	−0.040 99	−0.056 42	6	−0.012 59	+0.058 71	−0.046 14	−0.059 41
7	+ 0.010 77	+0.021 12	−0.031 90	−0.047 63	7	+0.006 13	+0.029 73	−0.035 86	−0.052 53
8	+ 0.020 48	+0.005 83	−0.026 29	−0.038 80	8	+0.018 68	+0.010 76	−0.029 46	−0.044 02
9	+ 0.027 11	−0.004 29	−0.022 81	−0.030 43	9	+0.027 43	−0.002 03	−0.025 41	−0.035 16
10	+ 0.031 87	−0.011 16	−0.020 72	−0.022 44	10	+0.033 75	−0.010 80	−0.022 94	−0.026 23
11	+ 0.035 48	−0.015 81	−0.019 66	−0.014 59	11	+0.038 44	−0.016 77	−0.021 67	−0.017 17
12	+ 0.038 27	−0.018 77	−0.019 50	−0.006 49	12	+0.041 94	−0.020 51	−0.021 43	+0.007 65
13	+ 0.040 40	−0.020 16	−0.020 25	+0.002 31	13	+0.044 40	−0.022 14	−0.022 25	+0.002 82
14	+ 0.041 76	−0.019 67	−0.022 10	+0.012 45	14	+0.045 64	−0.021 24	−0.024 41	+0.014 93
15	+ 0.041 87	−0.016 37	−0.025 50	+0.024 82	15	+0.044 91	−0.016 48	−0.028 43	+0.029 65
16	+ 0.039 55	−0.008 09	−0.031 45	+0.040 74	16	+0.040 45	−0.004 89	−0.035 56	+0.048 33
17	+ 0.031 89	+0.010 08	−0.041 96	+0.062 04	17	+0.028 13	+0.020 18	−0.048 30	+0.072 42
18	+ 0.011 82	+0.049 94	−0.061 74	+0.090 76	18	−0.001 69	+0.074 06	−0.072 36	+0.101 84
19	− 0.039 20	+0.142 08	−0.102 88	+0.125 25	19	−0.071 54	+0.193 50	−0.121 96	+0.126 43
20	− 0.170 93	+0.372 57	−0.201 62	+0.134 42	20	−0.227 49	+0.464 42	−0.236 94	+0.082 80
21	− 0.481 93	+0.970 66	−0.488 81	−0.115 23	21	−0.462 90	+1.001 54	−0.538 66	−0.342 30
22	− 0.306 63	+1.779 27	−1.472 52	−1.954 83	22	+0.326 63	+0.990 93	−1.317 50	−1.948 18
23	+ 6.049 55	−2.684 17	−3.365 26	−2.545 06	23	+4.092 27	−1.941 57	−2.150 75	−1.155 97

TABLE XVI.

MARS; DIRECT ACTION; DEVELOPMENT OF A-COEFFICIENTS.

M, *g′*	*A*		*B*		*C*		*D*	
	cos	sin	cos	sin	cos	sin	cos	sin
0 0	+0.2634	0.0000	+0.0962	0.0000	−0.3595	0.0000	+0.0004	0.0000
0 1	−0.0870	−0.0950	−0.0158	−0.0104	+0.1029	+0.1054	−0.0252	+0.0182
0 2	−0.0020	+0.0208	−0.0030	−0.0062	+0.0050	−0.0144	0.0000	−0.0014
0 3	+0.0072	−0.0056	−0.0016	+0.0048	−0.0056	+0.0006	+0.0080	+0.0068
0 4	−0.0108	−0.0044	+0.0082	+0.0070	+0.0026	−0.0026	+0.0006	−0.0142
0 5	+0.0026	+0.0104	−0.0046	−0.0126	+0.0020	+0.0023	−0.0146	+0.0104
0 6	+0.0024	0.0000	0.0000	0.0000	−0.0024	0.0000	+0.0100	0.0000
1− 7	+0.0024	+0.0014	−0.0004	−0.0022	−0.0020	+0.0008	+0.0100	−0.0023
1− 6	+0.0036	−0.0129	−0.0034	+0.0156	−0.0002	−0.0027	−0.0129	−0.0016
1− 5	−0.0144	+0.0050	+0.0098	−0.0064	+0.0046	+0.0015	−0.0030	+0.0198
1− 4	+0.0096	+0.0076	−0.0052	−0.0090	−0.0045	+0.0013	+0.0101	−0.0152
1− 3	−0.0022	−0.0143	+0.0002	+0.0096	+0.0020	+0.0048	+0.0011	+0.0025
1− 2	−0.0530	+0.0562	−0.0084	+0.0066	+0.0615	−0.0629	−0.0141	−0.0210
1− 1	+0.4997	+0.0012	+0.1051	−0.0023	−0.6049	+0.0011	+0.0004	+0.0972
0 0	−0.1201	−0.1314	−0.0316	−0.0280	+0.1516	+0.1593	+0.0019	−0.0005
1+ 1	−0.0026	+0.0319	−0.0049	−0.0001	+0.0075	−0.0317	−0.0026	+0.0004
1+ 2	+0.0064	−0.0042	+0.0002	+0.0009	−0.0066	+0.0033	+0.0062	−0.0003
1+ 3	−0.0084	−0.0039	+0.0076	+0.0071	+0.0007	−0.0031	+0.0045	−0.0091
1+ 4	+0.0014	+0.0077	−0.0050	−0.0088	+0.0035	+0.0011	−0.0159	+0.0195
1+ 5	+0.0024	+0.0014	−0.0004	−0.0022	−0.0020	+0.0008	+0.0100	−0.0023

TABLE XVI.—*Concluded.*

MARS; DIRECT ACTION; DEVELOPMENT OF A-COEFFICIENTS.

M, g'	*A*		*B*		*C*		*D*	
	cos	sin	cos	sin	cos	sin	cos	sin
2− 8	+0.0028	+0.0025	−0.0014	−0.0038	−0.0013	+0.0013	+0.0094	+0.0015
2− 7	+0.0046	−0.0154	−0.0020	+0.0169	−0.0026	−0.0016	−0.0217	−0.0147
2− 6	−0.0186	+0.0054	+0.0127	−0.0050	+0.0059	−0.0004	+0.0059	−0.0203
2− 5	+0.0134	+0.0105	−0.0102	−0.0129	−0.0033	+0.0024	+0.0128	−0.0096
2− 4	−0.0026	−0.0128	+0.0032	+0.0119	−0.0006	+0.0008	−0.0098	−0.0020
2− 3	−0.0190	+0.0169	−0.0134	+0.0176	+0.0324	−0.0345	+0.0102	−0.0034
2− 2	+0.5775	+0.0028	−0.1153	−0.0055	−0.4622	+0.0026	−0.0016	+0.2950
2− 1	−0.1498	−0.1646	−0.0258	−0.0248	+0.1757	+0.1893	+0.0235	−0.0256
2 0	−0.0041	+0.0481	−0.0054	+0.0090	+0.0095	−0.0573	+0.0047	−0.0038
2+ 1	+0.0078	−0.0044	+0.0002	−0.0027	−0.0080	+0.0070	+0.0040	+0.0075
2+ 2	−0.0066	−0.0036	+0.0078	+0.0062	−0.0011	−0.0028	−0.0041	−0.0112
2+ 3	+0.0004	+0.0054	−0.0044	−0.0050	+0.0040	−0.0004	−0.0068	+0.0068
2+ 4	+0.0028	+0.0025	−0.0014	−0.0038	−0.0013	+0.0013	+0.0094	+0.0015
3− 9	+0.0029	+0.0034	−0.0026	−0.0046	−0.0003	+0.0012	+0.0086	+0.0018
3− 8	+0.0059	−0.0172	−0.0014	+0.0168	−0.0045	+0.0004	−0.0237	−0.0173
3− 7	−0.0237	+0.0058	+0.0174	−0.0033	+0.0063	−0.0026	+0.0085	+0.0249
3− 6	+0.0186	+0.0144	−0.0164	−0.0168	−0.0021	+0.0025	+0.0158	−0.0133
3− 5	−0.0028	−0.0155	+0.0052	+0.0150	−0.0024	+0.0004	−0.0139	−0.0030
3− 4	+0.0099	−0.0146	−0.0236	+0.0321	+0.0136	−0.0174	+0.0297	−0.0153
3− 3	+0.5826	+0.0046	−0.2491	−0.0085	−0.3334	+0.0039	−0.0040	+0.3882
3− 2	−0.2006	−0.2207	+0.0269	+0.0297	+0.1736	+0.1910	+0.0898	−0.0863
3− 1	−0.0064	+0.0694	−0.0044	+0.0132	+0.0108	−0.0826	−0.0055	−0.0048
3 0	+0.0114	−0.0064	−0.0006	−0.0050	−0.0108	+0.0113	+0.0020	+0.0077
3+ 1	−0.0059	−0.0028	+0.0080	+0.0042	−0.0021	−0.0014	−0.0041	−0.0100
3+ 2	−0.0005	+0.0036	−0.0032	−0.0016	+0.0038	−0.0020	−0.0048	+0.0063
3+ 3	+0.0029	+0.0034	−0.0026	−0.0046	−0.0003	+0.0012	+0.0086	+0.0018
4−10	+0.0029	+0.0039	−0.0035	−0.0048	+0.0006	+0.0009	+0.0078	+0.0013
4− 9	+0.0076	−0.0188	−0.0024	+0.0159	−0.0052	+0.0029	−0.0246	−0.0183
4− 8	−0.0300	+0.0063	+0.0239	−0.0020	+0.0060	−0.0043	+0.0094	+0.0314
4− 7	+0.0250	+0.0189	−0.0240	−0.0202	−0.0010	+0.0013	+0.0190	−0.0407
4− 6	−0.0032	−0.0219	+0.0062	+0.0198	−0.0031	+0.0021	−0.0168	−0.0038
4− 5	+0.0294	−0.0348	−0.0316	+0.0428	+0.0021	−0.0079	+0.0444	+0.0325
4− 4	+0.5247	+0.0050	−0.2945	−0.0092	−0.2302	+0.0042	−0.0052	+0.3954
4− 3	−0.2464	−0.2718	+0.0905	+0.0972	+0.1559	+0.1745	+0.1605	−0.1425
4− 2	−0.0101	+0.1033	−0.0010	−0.0032	+0.0110	−0.1001	−0.0334	−0.0055
4− 1	+0.0168	−0.0104	−0.0016	−0.0052	−0.0152	+0.0155	+0.0017	+0.0082
4 0	−0.0060	−0.0021	+0.0074	+0.0012	−0.0014	+0.0010	−0.0033	−0.0091
4+ 1	−0.0012	+0.0020	−0.0014	+0.0014	+0.0026	−0.0034	−0.0077	+0.0060
4+ 2	+0.0029	+0.0039	−0.0035	−0.0048	+0.0006	+0.0009	+0.0078	+0.0013
5−11	+0.0030	+0.0042	−0.0042	−0.0042	+0.0012	0.0000	+0.0071	−0.0001
5−10	+0.0094	−0.020[illegible]	−0.0043	+0.0150	−0.0050	+0.0053	−0.0231	−0.0187
5− 9	−0.0364	+0.0072	+0.0316	−0.0024	+0.0050	−0.0049	+0.0101	+0.0383
5− 8	+0.0328	+0.0240	−0.0324	−0.0232	−0.0005	−0.0009	+0.0228	−0.0282
5− 7	−0.0034	−0.0310	+0.0064	+0.0264	−0.0029	+0.0047	−0.0211	−0.0019
5− 6	+0.0390	−0.0439	−0.0346	+0.0464	−0.0044	−0.0027	+0.0517	+0.0443
5− 5	+0.4335	+0.0041	−0.2804	−0.0072	−0.1530	+0.0032	−0.0054	+0.3500
5− 4	−0.2706	−0.2995	+0.1394	+0.1500	+0.1313	+0.1493	+0.2115	−0.1914
5− 3	−0.0146	+0.1452	+0.0046	−0.0378	+0.0099	−0.1074	−0.0758	−0.0080
5− 2	+0.0260	−0.0174	−0.0054	−0.0016	−0.0204	+0.0189	+0.0056	+0.0123
5− 1	−0.0068	−0.0017	+0.0054	−0.0018	+0.0012	+0.0035	−0.0027	−0.0071
5 0	−0.0018	+0.0011	+0.0012	+0.0032	+0.0007	−0.0042	−0.0032	+0.0064
5+ 1	+0.0030	+0.0042	−0.0042	−0.0042	+0.0012	0.0000	+0.0071	−0.0001
6−12	+0.0029	+0.0042	−0.0044	−0.0032	+0.0015	−0.0010	+0.0062	−0.0011
6−11	+0.1080	−0.0215	−0.0068	+0.0144	−0.0040	+0.0071	−0.0220	−0.0170
6−10	−0.0432	+0.0088	+0.0386	−0.0042	+0.0045	−0.0046	+0.0083	+0.0445
6− 9	+0.0418	+0.0297	−0.0404	−0.0256	−0.0014	−0.0041	+0.0282	−0.0394
6− 8	−0.0041	−0.0425	+0.0058	+0.0348	−0.0017	+0.0077	−0.0259	0.0000
6− 7	+0.0396	−0.0432	−0.0322	+0.0433	−0.0074	0.0000	+0.0544	+0.0527
6− 6	+0.3336	+0.0024	−0.2353	−0.0038	−0.0984	+0.0014	−0.0042	+0.2824
6− 5	−0.2706	−0.2998	+0.1642	+0.1783	+0.1063	+0.1216	+0.2345	−0.2093
6− 4	−0.0192	+0.1851	+0.0114	−0.0798	+0.0078	−0.1052	−0.1213	−0.0117
6− 3	+0.0388	−0.0274	−0.0138	+0.0063	−0.0251	+0.0211	+0.0136	+0.0195
6− 2	−0.0088	−0.0016	+0.0034	−0.0036	+0.0054	+0.0052	+0.0010	−0.0067
6− 1	−0.0024	+0.0009	+0.0040	+0.0032	−0.0016	−0.0041	−0.0039	+0.0070
6 0	+0.0029	+0.0042	−0.0044	−0.0032	+0.0015	−0.0010	+0.0062	−0.0011

C. ACTION OF JUPITER.

§ 39. The action of Jupiter being computed on the same general method as Venus and Mars, but being much simpler, no detailed explanation seems necessary. Six systems, which suffice to carry the coefficients to terms of the third order in the eccentricities, were deemed enough.

The principal numbers used or derived are shown in the following tables. The fundamental data in the first table were derived from Hill's *Tables of Jupiter.*

TABLE XVII.

ECLIPTIC COÖRDINATES OF JUPITER FOR THE 12 POINTS OF DIVISION.

i	Arg. 1.	f + Red. to Ecliptic (Table 37).			log. r (Table 60).	l			$l-\pi_{\prime}$			x	y	z
		°	′	″										
0	1062.6047	93	49	10	0.716 624	105	1	44	185	31	36	−5.1832	−0.5015	+0.0137
1	1423.6537	123	0	50	0.726 798	134	13	23	214	43	16	−4.3816	−3.0364	+0.0714
2	1784.7027	151	3	0	0.733 962	162	15	33	242	45	26	−2.4808	−4.8184	+0.1114
3	2145.7517	178	26	55	0.736 696	189	39	28	270	9	21	+0.0148	−5.4537	+0.1248
4	2506.8007	205	48	41	0.734 514	217	1	15	297	31	7	+2.5072	−4.8125	+0.1091
5	2867.8497	233	44	42	0.727 800	244	57	16	325	27	8	+4.4010	−3.0300	+0.0675
6	3228.8987	262	47	7	0.717 865	273	59	41	354	29	33	+5.1982	−0.5012	+0.0092
7	3589.9477	293	16	53	0.707 018	304	29	26	24	59	19	+4.6167	+2.1517	−0.0512
8	3950.9967	325	13	29	0.698 343	336	26	2	56	55	55	+2.7242	+4.1841	−0.0969
9	4312.0457	358	7	4	0.694 769	9	19	38	89	49	30	+0.0151	+4.9518	−0.1134
10	340.5067	31	3	59	0.697 634	42	16	32	122	46	24	−2.6983	+4.1912	−0.0948
11	701.5557	63	9	2	0.705 860	74	21	36	154	51	28	−4.5987	+2.1583	−0.0474

TABLE XVIII.

JUPITER; DIRECT ACTION; SPECIAL VALUES OF THE A-COEFFICIENTS FOR 6 SYSTEMS.

System 0.					System 1.				
i	A	B	C	D	i	A	B	C	D
0	+0.008 62	−0.004 22	−0.004 40	+0.001 56	0	+0.007 65	−0.003 80	−0.003 83	+0.000 60
1	+0.002 35	−0.001 17	−0.003 52	+0.005 25	1	+0.003 28	+0.000 13	−0.003 40	+0.004 86
2	−0.001 91	+0.004 50	−0.002 59	+0.002 20	2	−0.001 51	+0.004 27	−0.002 76	+0.002 97
3	−0.001 76	+0.003 72	−0.001 95	−0.001 04	3	−0.002 13	+0.004 33	−0.002 20	−0.000 70
4	+0.000 05	+0.001 53	−0.001 58	−0.002 26	4	−0.000 13	+0.001 95	−0.001 82	−0.002 52
5	+0.001 81	−0.000 39	−0.001 41	−0.001 81	5	+0.002 06	−0.000 46	−0.001 61	−0.002 06
6	+0.002 77	−0.001 37	−0.001 40	−0.000 34	6	+0.003 11	−0.001 55	−0.001 56	−0.000 19
7	+0.002 49	−0.000 95	−0.001 54	+0.001 55	7	+0.002 43	−0.000 76	−0.001 67	+0.001 93
8	+0.000 61	+0.001 30	−0.001 91	+0.002 84	8	+0.000 10	+0.001 87	−0.001 97	+0.002 81
9	−0.002 28	+0.004 86	−0.002 58	+0.001 50	9	−0.002 39	+0.004 87	−0.002 48	+0.000 81
10	−0.002 04	+0.005 63	−0.003 58	−0.003 77	10	−0.001 40	+0.004 56	−0.003 15	−0.003 68
11	+0.005 41	−0.000 95	−0.004 46	−0.005 89	11	+0.004 49	−0.000 75	−0.003 73	−0.004 95
System 2.					System 3.				
0	+0.007 67	−0.003 81	−0.003 85	−0.000 69	0	+0.008 73	−0.004 27	−0.004 46	−0.001 58
1	+0.004 63	−0.000 83	−0.003 81	+0.005 02	1	+0.005 47	−0.000 95	−0.004 51	+0.005 97
2	−0.001 42	+0.004 67	−0.003 25	+0.003 80	2	−0.002 06	+0.005 67	−0.003 60	+0.003 79
3	−0.002 45	+0.005 00	−0.002 55	−0.000 84	3	−0.002 28	+0.004 86	−0.002 58	−0.001 50
4	+0.000 12	+0.001 88	−0.002 00	−0.002 87	4	+0.000 61	+0.001 29	−0.001 90	−0.002 82
5	+0.002 46	−0.000 78	−0.001 68	−0.001 92	5	+0.002 47	−0.000 94	−0.001 53	−0.001 54
6	+0.003 09	−0.001 54	−0.001 55	+0.000 21	6	+0.002 74	−0.001 36	−0.001 39	+0.000 33
7	+0.002 01	−0.000 43	−0.001 58	+0.002 04	7	+0.001 79	−0.000 39	−0.001 40	+0.001 80
8	−0.000 15	+0.001 92	−0.001 76	+0.002 46	8	+0.000 06	+0.001 52	−0.001 58	+0.002 25
9	−0.002 08	+0.004 23	−0.002 15	+0.000 68	9	−0.001 76	+0.003 71	−0.001 95	+0.001 04
10	−0.001 49	+0.004 19	−0.002 70	−0.002 90	10	−0.001 92	+0.004 52	−0.002 60	−0.002 21
11	+0.003 18	+0.000 19	−0.003 36	−0.004 82	11	+0.002 36	+0.001 19	−0.003 55	−0.005 30

TABLE XVIII.—*Concluded.*

Jupiter; Direct Action; Special Values of the A-coefficients for 6 Systems.

System 4.					System 5.				
i	A	B	C	D	i	A	B	C	D
0	+0.010 47	−0.005 21	−0.005 26	−0.000 94	0	+0.010 36	−0.005 15	−0.005 21	+0.001 06
1	+0.004 29	+0.000 44	−0.004 73	+0.006 84	1	+0.002 55	+0.001 57	−0.004 12	+0.006 16
2	−0.002 58	+0.005 91	−0.003 33	+0.002 64	2	−0.002 34	+0.005 16	−0.002 82	+0.001 95
3	−0.001 75	+0.004 00	−0.002 25	−0.001 77	3	−0.001 58	+0.003 55	−0.001 97	−0.001 48
4	+0.000 68	+0.000 96	−0.001 65	−0.002 46	4	+0.000 41	+0.001 11	−0.001 51	−0.002 24
5	+0.002 16	−0.000 80	−0.001 35	−0.001 39	5	+0.001 88	−0.000 57	−0.001 30	−0.001 52
6	+0.002 51	−0.001 25	−0.001 25	+0.000 14	6	+0.002 51	−0.001 25	−0.001 26	−0.000 16
7	+0.001 90	−0.000 59	−0.001 31	+0.001 52	7	+0.002 20	−0.000 83	−0.001 37	+0.001 39
8	+0.000 43	+0.001 11	−0.001 54	+0.002 28	8	+0.000 72	+0.000 96	−0.001 68	+0.002 52
9	−0.001 61	+0.003 62	−0.002 02	+0.001 52	9	−0.001 79	+0.004 10	−0.002 31	+0.001 82
10	−0.002 41	+0.005 30	−0.002 90	−0.002 01	10	−0.002 64	+0.006 06	−0.003 42	−0.002 72
11	+0.002 68	+0.001 55	−0.004 23	−0.006 32	11	+0.004 44	+0.000 36	−0.004 80	−0.006 91

TABLE XIX.

Jupiter; Direct Action; Development of the A-coefficients.

Arg.	$10^6 A$		$10^6 B$		$10^6 C$		$10^6 D$	
LJ, g'	cos	sin	cos	sin	cos	sin	cos	sin
0 0	+1347	0	+1237	0	−2583	0	− 1	0
0 1	− 4	− 58	− 1	− 19	+ 4	+ 76	− 18	+ 3
1−2	− 7	− 77	+ 17	+ 86	− 12	− 10	+ 82	− 13
1−1	+1112	0	+ 358	0	−1470	0	0	+ 539
1 0	− 11	−211	− 16	−184	+ 28	+395	0	+ 2
1+1	− 3	0	− 1	+ 2	+ 5	0	0	+ 1
2−3	− 3	− 82	+ 8	+ 82	− 4	− 1	+ 83	− 5
2−2	+3878	+ 2	−3526	− 5	− 355	+ 2	− 3	+3699
2−1	− 7	−165	− 5	− 53	+ 12	+217	+ 53	− 2
2 0	− 16	+ 2	− 15	+ 3	+ 30	− 5	+ 1	− 11
	0	0	0	0	0	0	0	0
3−4	+ 13	− 40	− 12	+ 40	− 3	0	+ 42	+ 12
3−3	+1779	+ 1	−1700	− 3	− 79	+ 1	− 2	+1737
3−2	− 28	−672	+ 24	+595	+ 5	+ 77	+633	− 25
3−1	− 18	+ 3	− 6	+ 1	+ 24	− 4	0	− 6
4−5	+ 10	− 10	− 8	+ 10	0	+ 1	+ 21	+ 9
4−4	+ 587	− 3	− 570	+ 1	− 18	+ 1	+ 2	+ 575
4−3	− 18	−434	+ 16	+412	+ 2	+ 22	+423	− 17
4−2	− 80	+ 7	+ 70	− 6	+ 11	0	− 6	− 76
5−6	+ 2	+ 16	− 3	− 14	+ 1	− 1	+ 26	+ 6
5−5	+ 176	− 2	− 172	+ 2	− 3	0	+ 3	+ 154
5−4	− 8	−185	+ 7	+180	0	+ 5	+184	− 8
5−3	− 66	+ 6	+ 64	− 6	+ 4	− 1	− 6	− 67

D. ACTION OF SATURN AND MERCURY.

§ 40. The inequalities due to the direct action of Saturn are so minute that an approximate development will suffice. I have therefore used the development of Δ^{-3} and Δ^{-5} by spherical harmonics. We put a_1 for the mean distance of Saturn, L, as usual, for the difference of mean heliocentric longitudes of the planet and Earth ($L = \text{s} - g'$), and α for the ratio of the mean distances. With this notation the developments to α^4 are

$$\frac{a_1^3}{\Delta^3} = 1 + \tfrac{9}{4}\alpha^2 + \tfrac{225}{64}\alpha^4 + \cdots + (3 + \tfrac{45}{8}\alpha^2 + \cdots)\alpha \cos L$$
$$+ (\tfrac{15}{4} + \tfrac{105}{16}\alpha^2)\alpha^2 \cos 2L + \tfrac{35}{8}\alpha^3 \cos 3L + \tfrac{315}{64}\alpha^4 \cos 4L$$

$$\frac{a_1^5}{\Delta^5} = 1 + \tfrac{25}{4}\alpha^2 + \tfrac{1225}{64}\alpha^4 + \cdots + (5 + \tfrac{175}{8}\alpha^2 + \cdots)\alpha \cos L$$
$$+ (\tfrac{35}{4} + \tfrac{525}{16}\alpha^2 \cdots)\alpha^2 \cos 2L + \tfrac{105}{8}\alpha^2 \cos 3L + \tfrac{1155}{64}\alpha^4 \cos 4L$$

This development is valid when the eccentricities are taken account of, provided we use the true radii vectores and true longitudes instead of the mean ones. But this is unnecessary in the present case. For Saturn we have

$$\alpha = 0.1070$$

Reducing to numbers this gives

$$\frac{a_1^3}{\Delta^3} = 1.0262 + 0.328 \cos L + 0.044 \cos 2L + 0.005 \cos 3L \cdots$$

$$\frac{a_1^5}{\Delta^5} = 1.0741 + 0.557 \cos L + 0.110 \cos 2L + .016 \cos 3L + \cdots$$

For the geocentric coördinates X, Y, Z, of Saturn we have

$$X = a' - a_1 \cos L = a_1(\alpha - \cos L) \qquad Y = -a_1 \sin L \qquad Z = 0$$

Then

$$2K = \frac{a'^3}{\Delta^5}(X^2 - Y^2) = \alpha^3 \frac{a_1^5}{\Delta^5}(\alpha^2 - 2\alpha \cos L + \cos 2L)$$

$$C_1 = a'^3 C = -\frac{\alpha^3}{3}\frac{a_1^3}{\Delta^3}$$

$$D_1 = a'^3 D = \alpha^3 \frac{a_1^5}{\Delta^5}(\tfrac{1}{2} \sin 2L - \alpha \sin L)$$

Reducing to numbers, and performing the necessary multiplications we find

$$10^3 K = +.0027 + .031 \cos L + .620 \cos 2L$$

$$10^3 C_1 = -.419 - .135 \cos L - .018 \cos 2L$$

$$10^3 D_1 = -.029 \sin L + .607 \sin 2L$$

Then, the principal terms are

$$\begin{aligned}10^8MK &= +.0013 + .015 \cos L + .307 \cos 2L\\ 10^8MC_1 &= -\ \ .208 - .067 \cos L - .009 \cos 2L\\ 10^8MD_1 &= \cdots\cdots - .014 \sin L + .302 \sin 2L\end{aligned}$$

§ 41. The mass of Mercury is so minute that its action upon Venus, the only planet whose motion it can sensibly affect, has never been determined with certainty. There is every reason to believe that the uncertain determinations of the mass which have been made were too great by 2 or 3 times their entire amount. From Hill's estimate, based on the volume and probable density of the planet, it is very probable that the mass is less than 1 ÷ 10 000 000 that of the Sun. From the results of § 30 it is inferred that its secular effect on the motion of the lunar elements is proportionally yet smaller than its mass.

The only periodic inequalities that could become sensible are those of comparatively long period. Their probable limiting values are considered in *Action*, p. 273, from which it appears that the largest inequality is that depending on the argument

$$l + \pi + 3M' + g'$$

and that the limiting value of the coefficient was estimated at 0″.1. For another argument the limiting value was 0″.04. These estimates rest on a mass double of what may now be considered the most probable value. For these reasons it was intended to leave the action of Mercury entirely out of consideration in the present investigation. But, for the sake of completeness, and to leave open as few questions as possible, it was at length decided to compute the action in the same way as that of Venus. Twelve systems and twelve indices were used. With 144 special values, it is easy to compute not only the secular, but the principal periodic terms. Among the results are the following constant terms and terms depending on the above argument, the form being

$$A = A_0 + A_c \cos(3\mathrm{M}' + g') + A_s \sin(3\mathrm{M}' + g')$$

$$\begin{array}{lll} A_0 = +0.867 & A_c = -.00059 & A_s = 0\\ B_0 = -0.381 & B_c = +.00035 & B_s = +.00008\\ C_0 = -0.486 & C_c = +.00026 & C_s = -.00006\\ D_0 = +0.0022 & D_c = +.0005 & D_s = -.0023\\ K_0 = +0.624 & K_c = -.00047 & K_s = -.00004 \end{array}$$

§ 42. *K-coefficients.* From the preceding developments of A, B, C, and D for the four disturbing planets the coefficients $K = \frac{1}{2}\ (A - B)$ are formed, and K, C_1, and D_1 are multiplied by M.

This special set of coefficients, containing the factor M, are designated as K-coefficients. Their values are tabulated for Venus, Mars, and Jupiter as follows.

The values for Saturn are found at the end of § 40 preceding.

TABLE XX.

K-coefficients for Direct Action of Venus.

v, g'	$10^3\,MK_c$	$10^3\,MK_s$	$\frac{1}{2}10^3\,MC_c$	$\frac{1}{2}10^3\,MC_s$	$10^3\,MD_c$	$10^3\,MD_s$
	″	″	″	″	″	″
0, 0	+ 5.903	0.00	−3.406	0.000	0.000	0.000
0, 1	+ 0.44	−0.11	−0.30	+0.07	+0.03	+0.33
0, 2	+ 0.13	−0.14	−0.09	+0.10	+0.05	+0.06
1, −2	+ 0.71	+0.15	−0.36	−0.08	+0.08	−0.52
1, −1	+10.95	0.00	−6.03	0.00	−0.01	−4.92
1, 0	+ 0.17	−0.07	−0.22	+0.05	0.00	+0.22
2, −4	+ 0.16	+0.14	−0.08	−0.08	+0.09	−0.11
2, −3	+ 0.93	+0.19	−0.39	−0.08	+0.13	−0.73
2, −2	+10.75	0.00	−4.99	0.00	−0.01	−7.09
2, −1	+ 0.02	−0.04	−0.16	+0.03	−0.01	+0.22
3, −6	+ 0.02	+0.03	−0.01	−0.01	+0.02	−0.01
3, −5	+ 0.170	+0.147	−0.078	−0.075	+0.11	−0.13
3, −4	+ 1.08	+0.22	−0.38	−0.08	+0.17	−0.91
3, −3	+10.24	0.00	−4.00	0.00	−0.01	−7.91
4, −6	+ 0.18	+0.15	−0.07	−0.07	+0.11	−0.15
4, −5	+ 1.17	+0.24	−0.36	−0.08	+0.20	−1.04
4, −4	+ 9.42	0.00	−3.14	0.00	−0.01	−7.89
5, −8	+ 0.019	+0.028	−0.007	−0.012	+0.024	−0.017
5, −7	+ 0.19	+0.14	−0.06	−0.06	+0.12	−0.16
5, −6	+ 1.21	+0.24	−0.32	−0.07	+0.21	−1.05
5, −5	+ 8.40	0.00	−2.43	0.00	−0.01	−7.39
6, −7	+ 1.19	+0.24	−0.28	−0.06	+0.21	−0.11
6, −6	+ 7.32	0.00	−1.86	0.00	0.00	−6.64

TABLE XXI.

K-coefficients for Direct Action of Mars.

M, g'	$10^3\,MK_c$	$10^3\,MK_s$	$\frac{1}{2}10^3\,MC_c$	$\frac{1}{2}10^3\,MC_s$	$10^3\,MD_c$	$10^3\,MD_s$
	″	″	″	″	″	″
0, 0	+ 0.0465	0.000	−0.1006	0.000	0.000	0.000
0, 1	− 0.020	−0.024	+0.028	+0.029	−0.008	+0.010
1, −2	− 0.012	+0.014	+0.017	−0.018	−0.008	−0.012
1, −1	+ 0.110	+0.001	−0.169	0.000	0.000	+0.054
1, 0	− 0.025	−0.029	+0.042	+0.045	0.000	0.000
2, −3	− 0.002	0.000	+0.009	−0.010	+0.006	−0.002
2, −2	+ 0.194	+0.002	−0.129	+0.001	0.000	+0.165
2, −1	− 0.035	−0.039	+0.049	+0.053	+0.013	−0.014
2, 0	0.000	+0.011	+0.003	−0.016	+0.002	−0.002
3, −3	+ 0.233	+0.004	−0.093	+0.001	−0.002	+0.217
3, −2	− 0.064	−0.070	+0.049	+0.053	+0.050	−0.048
4, −4	+ 0.229	+0.004	−0.064	+0.001	−0.003	+0.221
4, −3	− 0.094	−0.103	+0.044	+0.049	+0.090	−0.083
4, −2	− 0.002	+0.030	+0.003	−0.028	−0.019	−0.003
5, −4	− 0.115	−0.126	+0.037	+0.042	+0.119	−0.107
5, −3	− 0.005	+0.051	+0.003	−0.030	−0.042	−0.004
5, −2	+ 0.009	−0.004	−0.006	+0.005	+0.003	+0.007
6, −5	− 0.122	−0.134	+0.030	+0.034	+0.131	−0.117
6, −4	− 0.009	+0.074	+0.002	−0.029	−0.068	−0.007
6, −3	+ 0.015	−0.009	−0.007	+0.006	+0.008	+0.011

TABLE XXII.

K-COEFFICIENTS FOR DIRECT ACTION OF JUPITER.

$j,\ g'$	$10^3 K_c$	$10^3 K_s$	$\frac{1}{2} 10^3 MC_c$	$\frac{1}{2} 10^3 MC_s$	$10^3 MD_c$	$10^3 MD_s$
	″	″	″	″	″	″
0, 0	+ 0.091	0.000	−2.135	0.000	0.000	0.000
0, 1	− 0.002	−0.032	+0.003	+0.063	−0.030	+0.005
1, −2	− 0.020	−0.135	−0.010	−0.008	+0.135	−0.021
1, −1	+ 0.623	0.000	−1.215	0.000	0.000	+0.593
1, 0	+ 0.004	−0.022	+0.023	+0.326	0.000	+0.003
1, +1	− 0.002	−0.001	+0.004	0.000	0.000	+0.002
2, −3	− 0.009	−0.135	−0.003	−0.001	+0.137	−0.008
2, −2	+ 6.119	+0.006	−0.293	+0.001	−0.005	+6.114
2, −1	− 0.001	−0.003	+0.010	+0.179	+0.087	−0.003
2, 0	− 0.001	−0.001	+0.025	−0.004	+0.001	−0.018
3, −3	+ 2.875	+0.003	−0.065	+0.001	−0.003	+2.875
3, −2	− 0.043	−1.048	+0.004	+0.064	+1.046	−0.041

CHAPTER V.

PLANETARY COEFFICIENTS FOR THE INDIRECT ACTION.

§ 43. Our next step is to form the coefficients G, J, and I, which are the planetary coefficients for the indirect action, and correspond to K, $\frac{1}{2}C$, and D. These we have found to be linear functions of the perturbations in the motion of the Earth around the Sun produced by the action of all the planets. From the way in which they are formed it will be seen that they should include all deviations in the motion of the Sun from the actual formulæ adopted for the expression of Ω as used in determining the action of the Sun itself. It would therefore be necessary, in strictness, to include the effect of any corrections that may be necessary to the elements of the Sun's motion employed by Delaunay. But as the eccentricity of the Earth's orbit enters as a symbolic quantity into the theories of both Delaunay and Brown, it will not be necessary to apply any correction on this account. The same remark applies to the position of the Earth's perihelion. But as the solar elements are assumed to be constant in the first integration it is necessary to take into account the effects of their secular variations, as well as of the periodic inequalities.

Moreover, in developing the action of the Sun upon the Moon for the first integration, it is assumed that the mean distance of the Earth's orbit is strictly connected with its mean motion by the fundamental relation

$$a^3n^2 = m' + \mu$$

It is therefore necessary to include in $\delta\rho'$ the constant correction arising from the action of the planets.

We may conveniently classify the various terms of $\delta v'$ and $\delta\rho'$ which are to be used in the expressions (60) as follows:

1. The terms arising from the secular variation of the eccentricity of the Earth's orbit.

2. Constant and periodic terms independent of the mean longitude of the disturbing planet.

3. Periodic terms containing that mean longitude.

§ 44. *Secular terms arising from the variation of the eccentricity of the Earth's orbit.*

The action of the Sun upon the Moon being a function of the eccentricity of the Earth's orbit it follows that the indirect action will vary with that element. The variation may be taken account of by assigning to $\delta v'$ and $\delta\rho'$ the increments of the

Earth's polar coördinates due to the variation of the eccentricity. It is not necessary to take into account the variation of the solar perigee, because this element is retained in its general form in the final expressions of all the perturbations.

To find the required values of $\delta v'$ and $\delta \rho'$ we differentiate the expressions for v' and ρ' in terms of the eccentricity, thus obtaining

$$\begin{aligned}\frac{\partial v'}{\partial e'} &= (2 - \tfrac{3}{4}e'^2)\sin g' + (\tfrac{5}{2}e' - \tfrac{11}{6}e'^3)\sin 2g' + \tfrac{13}{4}e'^2\sin 3g' + \tfrac{103}{24}e'^3\sin 4g' \\ \frac{\partial \rho'}{\partial e'} &= \tfrac{1}{2}e' + \tfrac{1}{8}e'^3 - (1 - \tfrac{9}{8}e'^2)\cos g' - (\tfrac{3}{2}e' - \tfrac{11}{6}e'^3)\cos 2g' - \tfrac{17}{8}e'^2\cos 3g' - \tfrac{71}{24}e'^3\cos 4g'\end{aligned} \qquad (84)$$

Putting $\Delta e'$ for the increment of e' due to secular variation, the values of $\delta v'$ and $\delta \rho'$ to quantities of the first order are found by multiplying these derivatives by $\Delta e'$. To determine what terms of higher order are necessary we remark that for an interval of 1000 years before or after 1900 we have

$$\Delta e' = \pm .000418 = \pm 86''.0$$

whence

$$(\Delta e')^2 = 0''.035$$

This quantity is so small that the powers of $\Delta e'$ above the first order may be dropped. But $\Delta e'$ will contain terms in T^2 which it will be well to include for the sake of approximation to rigor in the theory.

Substituting in the values of the differential coefficients just found the numerical value of e' for 1850,

$$e' = .0167711$$

we shall have

$$\begin{aligned}\frac{\partial v'}{\partial e'} &= 1.999\,79 \sin g' + .041\,92 \sin 2g' + .000\,91 \sin 3g' + .000\,02 \sin 4g' \\ \frac{\partial \rho'}{\partial e'} &= .008\,39 - .999\,68 \cos g' - .025\,15 \cos 2g' - .000\,60 \cos 3g' - .000\,01 \cos 4g'\end{aligned} \qquad (85)$$

The value of $\Delta e'$ by which these expressions are to be multiplied is that used in the author's *Tables of the Sun:*

$$\Delta e' = -8''.595\,T - 0''.0260\,T^2$$

T being counted in centuries from 1850.

The corresponding portions of G, J, and I are found by substituting for $\delta v'$ and $\delta \rho'$ in the expressions (60) the quantities

$$\partial v' = \frac{\partial v'}{\partial e'}\Delta e' \qquad \partial \rho' = \frac{\partial \rho'}{\partial e'}\Delta e'$$

If we suppose that G, J, and I are expressed in the form

$$G = G_0 + G_1 \Delta e'$$

with similar forms for J and I, we find by developing to e'^2

$$G_1 = +\tfrac{15}{4}e' - (\tfrac{9}{4} - \tfrac{549}{32}e'^2)\cos g' - \tfrac{51}{4}e' \cos 2g' - \tfrac{1269}{32}e'^2 \cos 3g'$$
$$J_1 = -\tfrac{3}{4}e' - (\tfrac{3}{4} + \tfrac{81}{32}e'^2)\cos g' - \tfrac{9}{4}e' \cos 2g' - \tfrac{159}{32}e'^2 \cos 3g'$$
$$I_1 = (3 - \tfrac{279}{16}e'^2)\sin g' + \tfrac{51}{4}e' \sin 2g' + \tfrac{883}{16}e'^2 \sin 3g'$$

The following numerical values have not been formed from these, but by multiplying the numerical values of the factors given in (60) and (85), which are derived from developments to e'^3.

$$\begin{aligned} G_1 &= +.06238 - 2.24517 \cos g' - .21296 \cos 2g' - .01115 \cos 3g' \\ J_1 &= -.01257 - 0.75039 \cos g' - .03771 \cos 2g' - .00140 \cos 3g' \\ I_1 &= + 2.96280 \sin g' + .21366 \sin 2g' + .04342 \sin 3g' \end{aligned} \qquad (86)$$

§ 45. *Terms independent of the mean longitude of the disturbing planet.* These terms arise from the terms of $\delta v'$ and $\delta\rho'$ which are either constants, or functions of g' alone. In the case of the longitude the eccentricity and perihelion of the Earth's orbit are so adjusted that both the constant terms and those dependent on Arg. g' shall vanish, leaving the only terms of $\delta v'$ to be considered those depending on Arg. $2g'$ etc. Both these terms themselves and the factors by which they are subsequently multiplied to form G, J, and I are so minute that the results are assumed to be insensible; we have, therefore, only to consider the terms of $\delta\rho'$ which remain after the adjustment of the eccentricity and perihelion just mentioned. These might be derived from the numbers in *Tables of the Sun;* but the author finds that the results have not been carried out with the precision desirable in the present problem. He has, therefore, computed these terms independently from theory, using the method of variation of elements, and carrying the results to terms of the second order in the eccentricities and mutual inclination. The general formulæ are as follows.* The accented quantities refer to the outer planet.

Action of an outer on an inner planet.

$$\delta\rho = m'a\{\rho_0 + \rho_1 \cos \Pi + (\rho_{0,c} + \rho_{1,c} \cos \Pi) \cos g + \rho_{1,s} \sin \Pi \sin g\}$$

where

$$\Pi = \pi - \pi'$$

$$\rho_0 = -\tfrac{1}{2}DA_0 + \frac{e^2}{48}(3D - 8D^2 - 2D^3)A_0 - \frac{e'^2}{24}(D^2 + D^3)A_0 \qquad \rho_1 = \frac{ee'}{24}(-7D + 5D^2 + 2D^3)A_1$$

$$\rho_{0,c} = \frac{e}{8}(3D + 2D^2)A_0 \qquad \rho_{1,c} = \frac{e'}{4}(1 - D^2)A_1 \qquad \rho_{1,s} = -\frac{e'}{4}(1 - D^2)A_1$$

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* The derivation has appeared in the *Astronomical Journal*, vol. xxv.

Action of an inner on an outer planet.

$$\delta\rho' = m\{\rho_0' + \rho_1' \cos \Pi' + (\rho_{0,c}' + \rho_{1,c}' \cos \Pi') \cos g' + \rho_{1,s} \sin \Pi' \sin g'\}$$

where

$$\Pi' = \pi' - \pi$$

$$\rho_0' = \tfrac{1}{8}(1 + D)A_0 - \frac{e'^2}{48}(9 + 13D + 2D^2 - 2D^3)A_0 + \frac{e^2}{24}(D + 2D^2 + D^3)A_0$$

$$\rho_1' = \frac{ee'}{24}(10 + 11D - D^2 - 2D^3)A_1 \qquad \rho_{0,c}' = \frac{e'}{8}(-1 + D + 2D^2)A_0$$

$$\rho_{1,c}' = -\frac{e}{4}(2D + D^3)A_1 \qquad \rho_{1,s}' = \frac{e}{4}(2D + D^2)A_1$$

The two actions are mutually interchanged by replacing D by $-(1 + D)$ in either. They were, however, developed independently in order that this relation might serve as a test of the accuracy of both.

The coefficients A_0 and A_1 are functions of the mutual inclination of the orbits ($\sigma = \sin \tfrac{1}{2}I$) and of the coefficients $b_s^{(i)}$ defined by the development

$$(1 - 2\alpha \cos L + \alpha^2)^s = \tfrac{1}{2}\Sigma b_s^{(i)} \cos iL$$

$$A_0 = b_{\frac{1}{2}}^{(0)} - \sigma\alpha^2 b_{\frac{3}{2}}^{(1)} \qquad A_1 = b_{\frac{1}{2}}^{(1)} - \tfrac{1}{2}\sigma\alpha^2(b_{\frac{3}{2}}^{(0)} + b_{\frac{3}{2}}^{(2)})$$

D^n means the nth derivative as to log a, or the symbolic value of $[a(\partial/\partial a)]^n$. If it be desired to use the usual successive derivatives as to a itself, we may do so by the substitutions

$$D = \alpha D_a \qquad D^2 = \alpha D_a + \alpha^2 D_a^2 \qquad D^3 = \alpha D_a + 3\alpha^2 D_a^2 + \alpha^3 D_a^3$$

From the numerical values of the coefficients A_0, A_1, and their D's we have the following results:

Action of Venus	$10^9\delta\rho' =$	$+ 1443.0$	$+ 31 \cos g' - 17 \sin g'$
Mars		$- 30.$	$+ 11 \cos g' - 8 \sin g'$
Jupiter		$- 1183.1$	$+ 90 \cos g' + 50 \sin g'$
Saturn		$- 55.4$	
Uranus		$- 1.0$	
Total		$+ 173.5$	$+ 132 \cos g' + 25 \sin g'$

Additional to these we have, for Mercury, with mass $= 10^{-7}$

$$10^9\delta\rho' = + 38.0 - 7 \cos g' + 3 \sin g'$$

which I treat separately.

From these (60) gives the following coefficients for G, J, and I, these quantities being expressed in the form

$$G = G_0 + G_c \cos g' + G_s \sin g'$$

Action of	$10^9 G_0$	$10^9 G_c$	$10^9 G_s$	$10^9 J_0$	$10^9 J_c$	$10^9 J_s$	$10^9 I_0$	$10^9 I_c$	$10^9 I_s$	
Venus	+3246.8	+233.3	− 38.2	+1083.3	+ 77.8	−12.8	+1.3		−217.7	
Mars	− 66.8	+ 21.4	− 18.0	− 22.3	+ 7.2	− 6.0	+ .6		+ 4.5	
Jupiter	−2655.0	+ 68.8	+112.3	− 886.0	+ 22.8	+37.5	−3.7	−.1	+178.2	
Saturn	− 124.6	− 6.3	0	− 41.6	− 2.1	0	0		+ 8.4	(87)
Uranus	− 2.2	− .1	0	− .8	0	0	0		+ .2	
Total	+ 397.6	+316.8	+ 56.1	+ 132.7	+105.5	+18.7	−1.9	−.1	− 26.6	
Mercury	85.1	− 11.4	+ 6.7	+ 28.4	− 3.9	+ 2.2	−0.2	0	− 5.7	

The totals here given are not formed by addition, but by an independent computation of the entire amount. Hence small discrepancies between the totals and the sums.

§ 46. *Periodic perturbations of the point G, containing the mean longitude of the disturbed planet.*

These are taken from *Astronomical Papers*, Vol. III, Part V, where they are found:

For Venus, on pp. 486–488
For Mars, " " 527–530
For Jupiter, " " 550–551

The perturbations by Venus are diminished by the factor .015 for reduction to the adopted value of the mass.

The expressions thus found are shown in tabular form below. In the original the constituents of the arguments were the mean anomalies alone, but, in the present work, the longitudes of the disturbing planets are reckoned from the Earth's perihelion. In order that the arguments for the direct and indirect inequalities may coincide, these perturbations have been transformed so that the planet's mean longitude shall be reckoned from the Earth's perihelion. The following are the numbers used:

		$\pi_4 - \pi'$
Earth	$\pi =$ 100° 21′	0° 0′
Venus	129 27	29 6
Mars	333 18	232 57
Jupiter	11 56	271 35

Then, if any pair of terms in $\delta v'$ or $\delta\rho'$ be represented by

$$v_c \cos(ig_4 + i'g') + v_s \sin(ig_4 + i'g')$$

we have to compute

$$b_c = v_c \cos i(\pi_4 - \pi') - v_s \sin i(\pi_4 - \pi')$$
$$b_s = v_c \sin i(\pi_4 - \pi') + v_s \cos i(\pi_4 - \pi')$$

to transform them into

$$b_c \cos(il_4 + i'g') + b_s \sin(il_4 + i'g')$$

l_4 being the mean longitude of the planet from π', designated by V, M, J, and S in the cases of the individual planets.

The original and transformed values of $\delta v'$ and $\delta\rho'$ are shown in Tables XXIII to XXVI.

The subsequent steps are shown in Tables XXVII–XXXIV in the following order:

The values of G, J, and I given in Tables XXVII–XXX are formed from the expressions of $\delta v'$ and $\delta\rho'$ in terms of g_4 by the formulæ (60). These are then subjected to the transformation of §46 and multiplied by the constant coefficient $10^8 m^2 = 5.595$. The factor 10^8 is introduced in order to have the most convenient unit in subsequent computation. As a check upon the work the values of J, G, and I were also computed using the transformed expressions for $\delta v'$ and $\delta\rho'$, and the results compared with the others. It has not been deemed necessary to set forth the steps of this simple duplicate computation.

TABLE XXIII.

ACTION OF VENUS ON THE EARTH.

Arg.	$\delta v'$		$\delta\rho'$		Arg.	$\delta v'$		$\delta\rho'$	
g_4, g'	cos	sin	cos	sin	V, g'	cos	sin	cos	sin
	″	″	″	″		″	″	″	″
−1, 0	+0.03	−0.01	−0.041	−0.018	−1, 0	0.00	−0.07	−0.045	+0.004
−1, 1	+2.35	−4.23	−0.980	−0.544	−1, 1	−0.01	−4.84	−1.121	+0.001
−1, 2	−0.06	−0.03	+0.032	−0.006	−1, 2	−0.06	0.00	+0.025	−0.021
−2, 1	−0.10	+0.06	+0.041	+0.065	−2, 1	0.00	+0.11	+0.077	−0.001
−2, 1	−4.70	+2.90	+1.709	+2.765	−2, 2	−0.02	+5.52	+3.251	+0.004
−2, 3	+1.80	−1.74	−0.282	−0.300	−2, 3	−0.53	−2.45	−0.404	+0.082
−2, 4	+0.03	−0.03	+0.018	+0.016	−2, 4	−0.01	−0.05	+0.023	−0.007
−3, 3	−0.67	+0.03	+0.020	+0.496	−3, 3	0.00	+0.67	+0.497	+0.003
−3, 4	+1.51	−0.40	−0.181	−0.689	−3, 4	−0.33	−1.53	−0.697	+0.149
−3, 5	+0.76	−0.68	+0.059	+0.069	−3, 5	−0.64	−0.79	+0.072	−0.056
−3, 6	+0.01	−0.01	+0.006	+0.006	−3, 6	−0.01	−0.01	+0.006	−0.006
−4, 4	−0.19	−0.09	−0.079	+0.160	−4, 4	0.00	+0.21	+0.178	0.000
−4, 5	−0.14	−0.04	−0.024	+0.089	−4, 5	+0.02	+0.15	+0.091	−0.018
−4, 6	+0.15	−0.04	−0.012	−0.043	−4, 6	−0.11	−0.11	−0.034	+0.030
−5, 6	−0.03	−0.02	−0.018	+0.020	−5, 6	+0.01	+0.04	+0.026	−0.006
−5, 7	−0.12	−0.03	−0.018	+0.065	−5, 7	+0.08	+0.09	+0.052	−0.044
−5, 8	+0.154	−0.001	0.000	−0.013	−5, 8	−0.128	−0.086	−0.007	+0.011
−8, 11	0.000	−0.002	−0.001	0.000	−8, 11	+0.002	+0.001	+0.001	−0.001
−8, 12	−0.008	−0.041	−0.020 5	+0.003 9	−8, 12	+0.038	+0.019	+0.009 3	−0.018 7
−8, 13	+1.268	+1.416	+0.004 23	−0.003 66	−8, 13	−1.895	+0.153	+0.000 36	+0.005 58
−8, 14	+0.021	+0.024	+0.010 6	+0.004 3	−8, 14	−0.032	+0.002	−0.009 8	+0.005 8

TABLE XXIV.

ACTION OF MARS ON THE EARTH.

Arg.	$\delta v'$		$\delta \rho'$		Arg.	$\delta v'$		$\delta \rho'$	
g_4, g'	cos	sin	cos	sin	M, g'	cos	sin	cos	sin
1, −1	−0″.216	−0″.167	−0″.043	+0″.056	1, −1	−0″.003	+0″.273	+0″.071	0″.000
1, 0	−0.008	−0.047	+0.013	−0.003	1, 0	−0.033	+0.034	−0.010	−0.008
2, −3	+0.040	−0.010	−0.006	−0.024	2, −3	−0.001	+0.041	+0.025	+0.001
2, −2	+1.963	−0.567	−0.272	−0.937	2, −2	+0.007	+2.043	+0.977	−0.005
2, −1	−1.659	−0.617	+0.030	−0.065	2, −1	+1.048	−1.427	+0.054	+0.047
2, 0	−0.024	+0.015	−0.008	−0.012	2, 0	−0.007	−0.027	+0.014	−0.005
3, −3	+0.053	−0.118	−0.073	−0.032	3, −3	+0.006	−0.129	−0.080	−0.004
3, −2	+0.396	−0.153	−0.037	−0.096	3, −2	+0.314	−0.286	−0.070	−0.077
4, −4	+0.001	+0.032	+0.022	−0.008	4, −4	+0.008	−0.033	−0.023	−0.005
4, −3	−0.131	+0.483	+0.219	+0.059	4, −3	+0.366	−0.342	−0.155	−0.165
4, −2	+0.526	−0.256	+0.021	+0.045	4, −2	−0.582	−0.059	+0.006	−0.049
5, −4	+0.049	+0.069	+0.041	−0.029	5, −4	−0.064	+0.055	+0.033	+0.038
5, −3	−0.038	+0.200	+0.041	+0.008	5, −3	−0.202	−0.020	−0.004	+0.042
6, −5	−0.020	−0.002	−0.001	+0.014	6, −5	−0.016	+0.012	+0.008	+0.011
6, −4	−0.104	−0.113	−0.048	+0.045	6, −4	−0.153	−0.014	−0.006	+0.065
6, −3	−0.011	+0.100	−0.013	−0.002	6, −3	+0.059	+0.081	−0.011	+0.008
15, −9	+0.018	−0.023	−0.008	−0.007	15, −9	−0.027	−0.011	−0.005	+0.010
15, −8	+0.201	−0.030	0.000	−0.003	15, −8	−0.083	−0.184	−0.003	+0.001

TABLE XXV.

ACTION OF JUPITER ON THE EARTH.

Arg.	$\delta v'$		$\delta \rho'$		Arg.	$\delta v'$		$\delta \rho'$	
g_4, g'	cos	sin	cos	sin	J, g'	cos	sin	cos	sin
1, −3	−0″.003	−0″.001	−0″.001	+0″.002	1, −3	−0″.001	+0″.003	+0″.002	+0″.001
1, −2	−0.155	−0.052	−0.037	+0.092	1, −2	−0.056	+0.154	+0.091	+0.040
1, −1	−7.208	+0.059	+0.026	+3.356	1, −1	−0.140	+7.207	+3.356	+0.067
1, 0	−0.307	−2.582	+0.108	−0.042	1, 0	−2.589	+0.236	−0.039	−0.109
1, +1	+0.008	−0.073	+0.037	+0.004	1, +1	−0.073	−0.010	+0.005	−0.037
2, −3	+0.011	+0.068	+0.049	−0.008	2, −3	−0.008	−0.069	−0.049	+0.005
2, −2	+0.136	+2.728	+1.910	−0.097	2, −2	+0.014	−2.731	−1.912	−0.008
2, −1	−0.537	+1.518	+0.654	+0.231	2, −1	+0.619	−1.486	−0.640	−0.267
2, 0	−0.022	−0.070	0.000	−0.004	2, 0	+0.018	+0.071	0.000	+0.004
3, −4	−0.005	+0.002	+0.001	+0.004	3, −4	−0.002	−0.005	−0.004	+0.001
3, −3	−0.162	+0.027	+0.021	+0.132	3, −3	−0.014	−0.164	−0.134	+0.010
3, −2	+0.071	+0.551	+0.378	−0.049	3, −2	−0.555	+0.025	+0.018	+0.381
3, −1	−0.031	+0.208	+0.082	+0.012	3, −1	−0.205	−0.048	−0.019	+0.081

TABLE XXVI.

ACTION OF SATURN ON THE EARTH.

Arg.	$\delta v'$		$10^9\delta\rho'$		Arg.	$\delta v'$		$\delta\rho'$	
g_4, g'	cos	sin	cos	sin	s, g'	cos	sin	cos	sin
1, −1	−0″.077	+0″.412	+972	+182	1, −1	−0″.003	+0″.419	+0″.204	+0″.001
1, 0	−0.003	−0.320	+ 18	− 2	1, 0	−0.060	−0.314	+0.004	−0.001
2, −2	+0.038	−0.101	−350	−132	2, −2	−0.001	−0.108	−0.077	0.000
2, −1	+0.045	−0.103	−236	−101	2, −1	+0.006	−0.113	−0.053	−0.002

TABLE XXVII.

PLANETARY COEFFICIENTS FOR THE INDIRECT ACTION OF VENUS.

Arg.	G		J		I	
g_4, g'	sin	cos	sin	cos	sin	cos
−1, 0	−0″.190	−0″.359	−0″.024	−0″.049	−0″.33	+0″.175
−1, 1	−1.220	−2.203	−0.408	−0.735	−6.34	+3.52
−1, 2	+0.074	+0.229	−0.014	+0.005	−0.125	−0.04
−2, 1	+0.533	+0.330	+0.101	+0.062	+0.32	−0.53
−2, 2	+6.113	+3.737	+2.069	+1.278	+4.26	−6.96
−2, 3	−0.755	−0.683	−0.173	−0.180	−2.63	+2.73
−2, 4	+0.108	+0.110	+0.007	+0.009	−0.10	+0.09
−3, 3	+1.001	+0.014	+0.359	+0.012	+0.016	−0.895
−3, 4	−1.590	−0.439	−0.507	−0.133	−0.62	+2.30
−3, 5	+0.191	+0.143	+0.039	+0.041	−1.02	+1.145
−3, 6	+0.056	+0.051	+0.005	+0.005	−0.045	+0.05
−4, 4	+0.372	−0.181	+0.122	−0.059	−0.13	−0.30
−4, 5	+0.190	−0.057	+0.069	−0.019	−0.06	−0.20
−4, 6	−0.099	−0.026	−0.030	−0.009	−0.06	+0.23
−5, 6	+0.061	−0.042	+0.011	−0.013	−0.03	−0.055
−5, 7	+0.136	−0.040	+0.049	−0.013	−0.04	−0.17
−5, 8	−0.031	0.000	−0.009	0.000	0.000	+0.231
−8, 12	−0.0550	+0.025	+0.002 90	−0.015	−0.007	+0.036
−8, 13	−0.0093	+0.0122	−0.002 60	+0.002 98	+2.126	+1.903
−8, 14	+0.073	−0.047	+0.003	+0.008	+0.093	+0.079

TABLE XXVIII.

Planetary Coefficient for the Indirect Action of Mars.

Arg.	*G*		*J*		*I*	
g_4 g'	sin	cos	sin	cos	sin	cos
	″	″	″	″	″	″
1, −1	+0.126	−0.098	+0.042	−0.033	−0.251	−0.324
1, 0	+0.015	+0.035	−0.001	+0.009	−0.073	−0.016
2, −3	−0.209	−0.057	−0.036	−0.009	−0.058	+0.208
2, −2	−2.026	−0.641	−0.704	−0.203	−0.871	+2.884
2, −1	−0.100	+0.082	−0.067	+0.017	−0.924	−2.485
2, 0	−0.113	+0.015	−0.010	−0.005	−0.003	−0.010
3, −3	−0.097	−0.174	−0.026	−0.056	−0.186	+0.101
3, −2	−0.215	−0.081	−0.072	−0.029	−0.227	+0.594
4, −4	−0.009	+0.085	−0.005	+0.020	+0.083	+0.008
4, −3	+0.111	+0.480	+0.045	+0.164	+0.714	−0.180
4, −2	+0.097	+0.035	+0.035	+0.020	−0.383	+0.788
5, −4	−0.063	+0.104	−0.022	+0.032	+0.114	+0.071
5, −3	+0.018	+0.091	+0.006	+0.032	+0.299	−0.057
6, −5	+0.039	−0.011	+0.012	−0.002	−0.011	−0.037
6, −4	+0.102	−0.104	+0.034	−0.036	−0.166	−0.156
6, −3	−0.006	−0.026	−0.001	−0.011	+0.150	−0.017
15, −9	−0.026	−0.020	−0.005	−0.006	−0.036	+0.035
15, −8	−0.006	+0.001	−0.003	0.000	−0.046	+0.301

TABLE XXIX.

Planetary Coefficients for the Indirect Action of Jupiter.

Arg.	*G*		*J*		*I*	
g_4 g'	cos	sin	cos	sin	cos	sin
	″	″	″	″	″	″
1, −3	−0.007	+0.039	−0.002	+0.006	− 0.039	−0.006
1, −2	−0.084	+0.759	−0.028	+0.132	− 0.757	−0.079
1, −1	−0.064	+7.554	+0.020	+2.518	−10.813	0.000
1, 0	+0.238	−0.266	+0.082	+0.032	− 0.479	−3.870
1, +1	+0.218	−0.012	+0.029	+0.004	− 0.007	−0.214
2, −3	+0.360	−0.030	+0.073	−0.008	+ 0.026	+0.352
2, −2	+4.407	−0.177	+1.447	−0.069	+ 0.167	+4.194
2, −1	+1.437	+0.522	+0.527	+0.171	− 0.808	+2.231
2, 0	−0.039	−0.023	+0.013	+0.001	− 0.036	−0.097
3, −4	+0.006	+0.024	+0.001	+0.005	− 0.023	+0.008
3, −3	+0.096	+0.290	+0.023	+0.098	− 0.236	+0.090
3, −2	+0.865	−0.109	+0.286	−0.035	+ 0.109	+0.839
3, −1	+0.177	+0.028	+0.069	+0.008	− 0.047	+0.304

TABLE XXX.

PLANETARY COEFFICIENTS FOR THE INDIRECT ACTION OF SATURN.

Arg.	G		J		I	
i, j s, g'	cos	sin	cos	sin	cos	sin
1, −1	+0″.443	+0″.005	+0″.153	+0″.001	−0″.006	+0″.616
1, 0	−0.001	−0.002	+0.007	−0.001	−0.090	−0.471
2, −2	−0.182	0.000	−0.059	0.000	+0.001	−0.171
2, −1	−0.118	−0.004	−0.041	−0.002	+0.009	−0.167

TABLE XXXI.

G-COEFFICIENTS FOR VENUS.

Arg.	10^3m^2G		10^3m^2J		10^3m^2I	
V, g'	cos	sin	cos	sin	cos	sin
−1, 0	− 2″.272	+0″.048	− 0″.304	+0″.016	− 0″.042	− 2″.089
−1, 1	−14.09	+0.031	− 4.703	+0.005	− 0.043	−40.572
−1, 2	+ 1.320	−0.261	− 0.014	−0.082	− 0.536	− 0.502
−2, 1	+ 3.508	+0.002	+ 0.663	+0.003	− 0.071	+ 3.510
−2, 2	+40.100	+0.253	+13.764	+0.121	− 0.263	+45.655
−2, 3	− 5.607	+1.024	− 1.354	+0.346	− 4.457	−20.735
−2, 4	+ 0.838	−0.205	+ 0.060	−0.022	− 0.209	− 0.723
−3, 3	+ 5.602	+1.186	+ 2.010	+0.028	− 0.146	+ 5.006
−3, 4	− 9.004	+2.036	− 2.869	+0.609	− 2.855	−13.017
−3, 5	+ 1.106	−0.749	+ 0.229	−0.219	− 5.400	− 6.668
−3, 6	+ 0.326	−0.270	+ 0.029	−0.027	− 0.239	− 0.292
−4, 4	+ 2.314	−0.018	+ 0.759	−0.008	+ 0.095	+ 1.827
−4, 5	+ 1.094	−0.180	+ 0.393	−0.077	+ 0.197	+ 1.152
−4, 6	− 0.431	+0.378	− 0.128	+0.121	− 0.873	− 1.003
−5, 6	+ 0.389	−0.148	+ 0.095	−0.010	+ 0.159	+ 0.312
−5, 7	+ 0.616	−0.500	+ 0.215	−0.185	+ 0.657	+ 0.723
−5, 8	− 0.098	+0.143	− 0.029	+0.042	− 1.065	− 0.732
−8, 11	+ 0.016	−0.011	+ 0.005	−0.006	− 0.788	+ 1.039
−8, 12	+ 0.1611	+0.2994	+ 0.039	−0.0777	− 0.091	+ 0.184
−8, 13	+ 0.0045	+0.0661	+ 0.001 45	+0.022 05	−15.912	+ 1.289
−8, 14	− 0.166	−0.4588	− 0.040	+0.024 6	− 0.681	+ 0.037

TABLE XXXII.

G-coefficients for Mars.

Arg.	10^3m^2G		10^3m^2J		10^3m^2I	
M, g'	cos	sin	cos	sin	cos	sin
	″	″	″	″	″	″
1, −1	+ 0.893	+0.012	+0.300	+0.006	−0.065	+ 2.293
1, 0	− 0.185	−0.105	−0.035	−0.036	−0.272	+ 0.318
2, −3	+ 1.211	+0.013	+0.207	+0.007	−0.006	+ 1.208
2, −2	+12.777	−0.089	+4.100	−0.014	+0.073	+17.531
2, −1	+ 0.413	+0.595	+0.335	+0.194	+1.161	+14.821
2, 0	− 0.309	0.000	+0.062	−0.012	−0.019	+ 0.129
3, −3	− 1.205	−0.155	−0.344	−0.022	+0.151	− 1.175
3, −2	− 0.857	−0.958	−0.296	−0.318	+2.641	− 2.384
4, −4	− 0.431	−0.208	−0.109	−0.035	+0.207	− 0.418
4, −3	− 1.956	−1.943	−0.638	−0.695	+2.961	− 2.864
4, −2	+ 0.119	−0.565	+0.009	−0.224	−4.877	− 0.502
5, −4	+ 0.404	+0.548	+0.139	+0.167	−0.600	+ 0.452
5, −3	− 0.054	+0.516	−0.018	+0.181	−1.695	− 0.166
6, −4	− 0.047	+0.815	−0.021	+0.276	−1.271	− 0.100
6, −3	− 0.130	+0.073	−0.050	+0.038	+0.495	+ 0.685
15, −9	− 0.109	+0.147	−0.018	+0.041	−0.246	− 0.135
15, −8	− 0.035	+0.003	−0.016	+0.005	−0.704	− 1.551

TABLE XXXIII.

G-coefficients for Jupiter.

Arg.	10^3m^2G		10^3m^2J		10^3m^2I	
J, g'	cos	sin	cos	sin	cos	sin
	″	″	″	″	″	″
1, −3	+ 0.218	+0.039	+ 0.022	+0.011	− 0.039	+ 0.217
1, −2	+ 4.240	+0.594	+ 0.738	+0.173	− 0.553	+ 4.241
1, −1	+42.237	+1.534	+14.088	+0.274	− 1.672	+60.471
1, 0	− 1.459	−1.376	+ 0.196	−0.458	−21.709	+ 2.097
1, +1	− 0.040	−1.225	+ 0.017	−0.168	− 1.203	+ 0.006
2, −3	− 2.012	+0.050	− 0.408	+0.022	− 0.061	− 1.991
2, −2	−24.668	−0.358	− 8.096	−0.061	+ 0.363	−23.488
2, −1	− 7.872	−3.363	− 2.887	−1.119	+ 5.192	−12.225
2, 0	+ 0.218	+0.140	− 0.067	−0.011	+ 0.173	+ 0.542
3, −4	− 0.140	+0.017	− 0.034	+0.006	− 0.017	− 0.129
3, −3	− 1.677	+0.403	− 0.559	+0.078	− 0.397	− 1.365
3, −1	+ 0.212	+4.874	+ 0.061	+1.611	− 4.733	+ 0.212
3, −1	− 0.240	+0.984	− 0.078	+0.380	− 1.672	− 0.402

TABLE XXXIV.

G-coefficients for Saturn.

Arg.	10^3m^2G		10^3m^2J		10^3m^2I	
S, g'	cos	sin	cos	sin	cos	sin
	″	″	″	″	″	″
1, −1	+2.48	+0.03	+0.86	+0.01	−0.03	+3.45
1, 0	+0.01	−0.01	+0.03	+0.01	−0.50	+2.64
2, −2	−1.02	0.00	−0.33	0.00	0.00	−0.96
2, −1	−0.66	−0.02	−0.23	−0.01	+0.05	−0.94

log a. The outcome of these considerations has been that, for the sake of trial and comparison, both Delaunay's and Brown's developments have been to a large extent independently used, and the results compared with a view of facilitating an estimate of the errors to which the analytic development is subject.

§ 48. *Reduction of coördinates to the radius vector of the Mean Sun as X-axis.*

In the developments in *Action* the Sun's perihelion was taken as the origin from which longitudes were measured. When the present work was undertaken, it being found that the development of the A-coefficients would be most easily effected by taking the direction of the mean Sun as the axis of X, the same origin had to be taken for the lunar coördinates. This has been done throughout the work; and it must be understood in the subsequent developments that x and y are referred to the radius vector of the mean Sun as axis of X except in terms arising from the motion of the ecliptic.

In the use of either theory we take, as the initial data of the problem, the rectangular coördinates of the Moon referred to the mean Moon as the axis of X, which coördinates we represent by x_1 and y_1. These coördinates are those which Brown's theory gives in the first instance, and they are also those which I have developed in powers of m, etc., from Delaunay's theory in *Action*, pp. 167 and 169. The notation of arguments from the latter paper is:

- g, the Moon's mean anomaly; g', the Sun's mean anomaly;
- λ, the mean elongation of the Moon from the Moon's ascending node, equivalent to Delaunay's F, or the mean argument of latitude;
- λ', the same for the mean Sun;
- θ, the longitude of the Moon's node;
- $l = g + \pi + \theta$, the Moon's mean longitude; $l' = g' + \lambda' + \theta$, the Sun's mean longitude;
- $D = l - l' = \lambda - \lambda'$, as in Delaunay, the mean Moon's departure from the mean Sun.

Putting δv for the excess of the true longitude over l, we then have

$$\begin{aligned} x_1 &= r\cos\beta\cos\delta v = a\Sigma k\cos N \\ y_1 &= r\cos\beta\sin\delta v = a\Sigma k'\sin N \\ z &= r\sin\beta = a\Sigma c\sin N' \end{aligned} \qquad (2)$$

where k, k', and c of dimensions 0 are developed in powers of e, e', y, and m.

The general form of the arguments N and N' is

$$N \text{ or } N' = ig + i'g' + j\lambda + j'\lambda' \qquad (3)$$

where $g = l - \pi$, $g' = l' - \pi'$, $\lambda = l - \theta$, $\lambda' = l' - \theta$.

The equations for transforming x_1 and y_1 into x and y are

$$x = x_1\cos D - y_1\sin D \qquad y = x_1\sin D + y_1\cos D \qquad (4)$$

$$x = x_0 \cos l' + y_0 \sin l' \qquad y = y_0 \cos l' - x_0 \sin l'$$

or

$$x_0 = x \cos l' - y \sin l' \qquad y_0 = y \cos l' + x \sin l' \tag{10}$$

The special term (9) will, therefore, transform into the terms of x and of y

$$x = h_0 \cos (N - l') \qquad y = h_0 \sin (N - l') \tag{11}$$

For the special functions required in the lunar theory we shall have the following transformations of the same form as (6)

$$\left.\begin{aligned} x^2 - y^2 &= (x_0^2 - y_0^2) \cos 2l' + 2x_0 y_0 \sin 2l' \\ 2xy &= 2x_0 y_0 \cos 2l' - (x_0^2 - y_0^2) \sin 2l' \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} x_0^2 - y_0^2 &= (x^2 - y^2) \cos 2l' - 2xy \sin 2l' \\ 2x_0 y_0 &= 2xy \cos 2l' + (x^2 - y^2) \sin 2l' \end{aligned}\right\} \tag{13}$$

The transformation of any one term may be made by the equations (6) by writing $+ 2l'$ or $- 2l'$ for $2D$.

If, as in most of the present work, the solar perigee is taken as the fundamental fixed X-axis, we write g' instead of l' in these equations.

An important remark to be made on these transformations of terms from one axis to the other is that the equality of coefficients expressed in the equations (9) and (11) is true only when the fixed axis of X is unrestricted. If, as will sometimes be more convenient, we take the direction of the solar perigee for this axis, some values of argument N in (9) will be equal with opposite signs. By combining the terms depending on these arguments the equality in question will cease to hold. If, however, the Sun's eccentricity is dropped, the general equations will remain valid for the Sun's perigee also.

It thus happens that, in the developments given in *Action*, pages 213–215, the coördinates are quite general, while the expressions for their squares and products given on pp. 217–223 are not general, because the solar perigee is here taken as the fundamental axis.

§ 50. Recalling that throughout the work we use the symbol D to represent the logarithmic derivative as to a of any function, a serious question is that of determining the value of this derivative with the necessary precision in each special class of terms. In actually performing the work so many tentative combinations have been made, as better and better methods were found, that it is difficult to present any one process as the definitive one. The following method was at length seen to be the best under the circumstances I have described. Let

$$u = a^i \phi(m)$$

be any function of the coördinates of which D is to be formed. Practically i will be equal to 1 or 2 according as the expression we are dealing with is of the first or second degree in the rectangular coördinates. If we can compute the value of $D\phi(m)$ with sufficient precision the complete value of Du will be

$$Du = ia^i\phi(m) + a^i D\phi(m) \tag{14}$$

If it is developed in powers of m, $\phi(m) = \alpha_0 + \alpha_1 m + \alpha_2 m^2 + \cdots$ and we shall have

$$D\phi(m) = \tfrac{3}{2}\alpha_1 m + 3\alpha_2 m^2 + \cdots$$

the coefficient of each term being $\frac{3}{2}$ of the exponent of m. (*v.* § 12, Eq. 23.)

If we have the numerically accurate value of any $\phi(m)$ from Brown's theory and an approximate one from the analytic development, the comparison of the two will furnish a rude index to the probable value of the omitted powers of m in the development. It follows that the nearest approximation to the value of Du will be obtained by using in the first term of the second member of (14) the numerical value of $a^i\phi(m) = u$, the analytic development being used only for the second term. Moreover, having an approximate estimate of the value of the omitted terms of the analytic development of the second member of (14), we may use it to correct the last term of this member. I conceive that no lack of theoretical rigor pertaining to this process will lead to an error of the slightest importance in the present work.

§ 51. *Formation of the D's from Delaunay's Theory.*

In the final formation of the D-derivatives I have extended the developments given in *Action*, by the aid of Delaunay's results, as follows. Delaunay expresses the reciprocal of the Moon's radius vector in a form which we may write

$$\frac{a}{r} = 1 + \pi_1$$

where π_1 is put for the sum of an infinite series of terms, each developed in powers of m, as well as of e, e', and γ. This quantity π_1 is related to the Moon's parallax π by the equation

$$\sin\pi = \frac{a_1}{a}(1 + \pi_1)$$

a_1 being the Earth's equatorial radius.

It is to be remarked that Delaunay's expression for the parallax was only carried to terms of the fifth order, so that it does not suffice for all theoretical purposes. It is indeed fairly probable that it would suffice for the object now in view. In order, however, to lessen the danger of any insufficiency in this respect I have, in forming the value of π_1, compared each coefficient in the expression of Delaunay's parallax found in my transformation of Hansen's lunar theory with the more accurate value derived from Hansen's or Brown's expression. We may conceive that the correction necessary to reduce Delaunay's coefficient to Hansen's value is of the form

$$\delta\pi_1 = \alpha_i m^i + \alpha_{i+1} m^{i+1} + \cdots$$

in which i is the power of m next above the highest to which Delaunay has carried his coefficient. From what we have already shown it follows that the corresponding correction to $D\pi_1$ is

$$\tfrac{2}{3} i a_i m^i + \tfrac{2}{3}(i + 1) a_{i+1} m^{i+1} \tag{15}$$

an approximate value of which is

$$\tfrac{2}{3} i \delta\pi_1$$

In order to make this correction rigorously exact we should know the values of the coefficients of the omitted powers of i. This being unknown, the minute correction is to a certain extent a matter of estimation. I do not conceive, however, that the uncertainty is at all important in the present investigation.

We have next to consider the D-derivatives of the three functions

$$\rho^2 - 3\zeta^2; \quad \xi_1^2 - \eta_1^2; \quad \text{and} \quad 2\xi_1\eta_1$$

Starting with the equations (2) the values of δv and β, developed in powers of m, are given at the end of Delaunay's *Theorie*, Vol. II. The values of $D\delta v$ are formed from these with great facility by means of the form (23) of § 12, because Delaunay gives the numerical value of each part of every term of the longitude.

The steps of the subsequent process consist in simple trigonometric multiplications, and are presented in tabular form on the following pages. The fundamental quantities are

$$\pi_1 = \frac{a}{r} - 1, \quad \delta v, \quad \beta \qquad \text{and} \qquad D\frac{a}{r}, \quad D\delta v, \quad D\beta$$

which are formed from Delaunay's numbers in the way just shown.

The following functions are then formed by trigonometric multiplication

$$\rho = 1 - \pi_1 + \pi_1^2 - \pi_1^3 + \cdots \quad \rho^2 = 1 - 2\pi_1 + 3\pi_1^2 - 4\pi_1^3 + \cdots \quad \rho^3 = 1 - 3\pi_1 + 6\pi_1^2 - 10\pi_1^3 + \cdots$$

In the final work, however, ρ has been formed from Brown's theory. Then

$$\rho^2 = \xi^2 + \eta^2 + \zeta^2$$

$$\sin\beta = \beta - \tfrac{1}{6}\beta^3 \qquad \cos\beta = 1 - \tfrac{1}{2}\beta^2 + \tfrac{1}{24}\beta^4 \qquad \cos^2\beta = 1 - \sin^2\beta$$

$$\sin\delta v = \delta v - \tfrac{1}{6}\delta v^3 \qquad \cos\delta v = 1 - \tfrac{1}{2}\delta v^2$$

$$\xi_1^2 - \eta_1^2 = \rho^2\cos^2\beta(1 - 2\sin^2\delta v) \qquad \xi_1\eta_1 = \rho^2\cos^2\beta\cos\delta v\sin\delta v$$

$$\zeta = \rho\sin\beta \qquad \zeta^2 = \rho^2\sin^2\beta \qquad D\cdot\rho^2 = -2\rho^3 D\pi_1$$

$$D\sin\beta = \cos\beta D\beta \qquad D\sin^2\beta = 2\sin\beta D\sin\beta \qquad D\rho^2\cos^2\beta = \cos^2\beta D\rho^2 + \rho^2 D\cos^2\beta \tag{16}$$

$$D\sin^2\delta v = 2\sin\delta v D\sin\delta v = 2\sin\delta v\cos\delta v D\delta v$$

$$D(\xi_1^2 - \eta_1^2) = (1 - 2\sin^2\delta v) D\rho^2\cos^2\beta - 2\rho^2\cos^2\beta D\sin^2\delta v$$

$$D\cdot\xi_1\eta_1 = \sin\delta v\cos\delta v D\rho^2\cos^2\beta + \rho^2\cos^2\beta(1 - 2\sin^2\delta v) D\delta v$$

$$D\cdot\zeta^2 = \rho^2 D\sin^2\beta + \sin^2\beta D\cdot\rho^2$$

$$\frac{\partial u}{\partial e}=\frac{\partial u}{\partial \mathrm{e}}\frac{\partial \mathrm{e}}{\partial e}+\frac{\partial u}{\partial \mathrm{k}}\frac{\partial \mathrm{k}}{\partial e}=1.9932\frac{\partial u}{\partial \mathrm{e}}-.0099\frac{\partial u}{\partial k} \qquad (17)$$

$$\frac{\partial u}{\partial \gamma}=\frac{\partial u}{\partial \mathrm{e}}\frac{\partial \mathrm{e}}{\partial \gamma}+\frac{\partial u}{\partial \mathrm{k}}\frac{\partial \mathrm{k}}{\partial \gamma}=0.9956\frac{\partial u}{\partial \mathrm{k}}-.0025\frac{\partial u}{\partial \mathrm{e}}$$

These equations enable us to find the derivatives as to e and γ from Brown's as to e and k.

§ 53. *Tables of the functions and derivatives of the Moon's coördinates.* The numerical processes by which the required functions of the coördinates were developed may be followed and tested by the aid of the following tables. The notation of the arguments, expressed by the indices in the first column, has been defined in § 48.

Owing to the circumstances mentioned in § 46, and to the widely different degree of precision required in the coefficients of different arguments, the numbers of these tables are not always consistently continuous. The terms of many, perhaps more than half the arguments, lead to no sensible inequalities; with these pains were not always taken to reach a higher degree of precision than was required to show the order of magnitude of the results. In the preliminary steps of the investigation it was deemed sufficient to carry the expressions for the Moon's coördinates to the 5th place of decimals, and those for the derivatives to the 3d or 4th place. But when the inequalities of the elements themselves were reached by integration, it was found that this degree of precision, while more than sufficient for the periodic terms in general, was not sufficient either in the terms related to the evection, or in those determining the secular variations and accelerations of l, π, and θ. A number of successive revisions was found to be necessary, in which the coefficients depending on the argument g' were carried to the 7th place of decimals. As the last place was always more or less doubtful only the sixth place has been included in the printing.

It may also be remarked that in commencing the tables it was supposed that the analytic development in *Action* would suffice for the work. This expectation proving ill-founded, the developments of the Moon's coördinates given by Delaunay, then those by Hansen, and finally those by Brown were successively used in the case of those terms in which greater precision was needed. Finally the D-derivatives were, in their important terms, recomputed by formulæ proposed by Dr. Ross, which were much briefer than those already given in (16) of § 51.

The want of homogeneity thus arising in the tables could be cured by a fresh lopment from the fundamental data of Brown and Delaunay, but I do not think nge would thus result in the expressions for the inequalities o Part IV.

TABLE XXXV.

FUNCTIONS OF THE LATITUDE AND THEIR DERIVATIVES.

Arg.				β	$D\beta$	$\sin\beta$	$D\sin\beta$	$\zeta=\rho\sin\beta$	$D\zeta$	$\partial\zeta/\partial e$	$\partial\zeta/\partial\gamma$
g	g'	λ	λ'	sin	sin	sin	sin	sin	sin	sin	sin
−2	0	1	0	−.000 154	+.000 09	−.000 154	+.000 90	−.000 088	+.000 08	−.004 9	−0.003 0
−1	0	1	0	−.004 842	+.000 32	−.004 840	+.000 32	−.007 240	+.000 44	−.130 4	−0.161 5
0	0	1	0	+.089 503	.000 00	+.089 413	.000 00	+.089 474	−.000 18	−.004 9	+1.998
1	0	1	0	+.004 898	−.000 05	+.004 896	−.000 05	+.002 466	+.000 03	+.045 4	+0.055
2	0	1	0	+.000 301	−.000 01	+.000 301	−.000 01	+.000 101	−.000 01	————	+0.002 3
0	−1	1	0	+.000 023	−.000 038	+.000 023	−.000 038	+.000 029	−.000 017	+.000 052	+0.000 642
1	−1	1	0	————	————	————	————	+.000 017	————	+.000 32	+0.000 39
−1	−1	1	0	————	————	————	————	−.000 042	————	−.000 76	−0.000 94
0	1	1	0	−.000 032	+.000 013	−.000 030	+.000 013	−.000 024	+.000 031	−.000 079	−0.000 535
0	0	3	0	−.000 030	+.000 01	−.000 030	+.000	————	————	————	————
−1	0	3	−2	+.000 968	+.002 02	+.000 968	+.002 01	+.000 545	+.001 24	+.009 7	+0.012 2
−1	−1	3	−2	————	————	————	————	+.000 026	————	+.000 47	————
0	0	3	−2	+.000 568	+.001 80	+.000 568	+.001 80	+.000 177	+.000 58	————	+0.004 1
−1	−1	1	−2	————	————	————	————	+.000 051	————	−.000 94	−0.001 16
0	−1	1	−2	+.000 144	+.000 30	+.000 144	+.000 30	+.000 165	+.000 18	————	+0.003 2
−1	0	1	−2	+.000 808	+.001 75	+.000 808	+.001 74	+.001 179	+.002 44	+.021 1	+0.026 1
0	0	1	−2	+.003 022	+.005 42	+.003 020	+.005 40	+.003 308	+.006 47	−.000 7	+0.073 9
1	0	1	−2	+.000 161	+.000 28	+.000 161	+.000 28	+.000 080	+.000 12	————	+0.001 7
0	1	1	−2	−.000 060	−.000 12	−.000 060	−.000 12	−.000 061	+.000 18	————	−0.001 4

TABLE XXXVI.

FUNCTIONS OF THE RADIUS VECTOR AND LATITUDE.

Arg.				$D\pi_1$	$\rho^2=r^2/a^2$	ζ^2	$\rho^2\cos^2\beta$	$D.\,\rho^2$	$D.\,\zeta^2$	$D.\,\rho^2\cos^2\beta$
g	g'	λ	λ'	cos	cos	cos	cos	cos	cos	cos
0	0	0	0	+.002 638	+1.002 866	+.004 038	+.998 828	−.004 43	+.000 01	−.004 44
1	0	0	0	−.001 40	−0.108 58	−.000 42	−.108 17	+.004 80	+.000 06	+.004 74
2	0	0	0	−.000 16	−0.001 52	−.000 02	−.001 51	+.000 08	+.000 01	+.000 08
−1	1	0	0	+.000 917	−0.000 70	−.000 00	−.000 70	−.001 77	.000 00	−.001 76
0	1	0	0	−.000 267	+0.000 266	+.000 001	+.000 266	+.000 713	+.000 001	+.000 712
1	1	0	0	−.000 637	+0.000 54	.000 00	+.000 54	+.001 27	.000 00	+.001 26
−1	0	2	0	+.000 10	+0.000 32	+.000 65	−.000 33	−.000 19	−.000 03	−.000 15
0	0	2	0	+.000 06	−0.000 02	−.003 98	+.003 97	−.000 10	+.000 02	−.000 12
1	0	2	0	+.000 00	+0.000 00	−.000 22	+.000 22	.000 00	.000 00	+.000 01
−2	0	2	−2	−.000 32	+0.001 83	.000 00	+.001 82	+.003 80	+.000 02	+.003 77
−1	0	2	−2	+.019 66	−0.018 91	−.000 03	−.018 88	−.035 22	−.000 06	−.035 15
0	0	2	−2	+.026 43	−0.014 86	−.000 29	−.014 57	−.049 46	−.000 54	−.048 92
1	0	2	−2	+.002 91	−0.000 48	−.000 02	−.000 46	−.001 54	−.000 03	−.001 50
2	0	2	−2	+.000 27	−0.000 04	.000 00	−.000 04	−.000 07	.000 00	−.000 07
−1	1	2	−2	+.000 216	+0.000 06	.000 00	+.000 06	−.000 48	.000 00	−.000 47
0	1	2	−2	−.000 441	+0.000 144	+.000 005	+.000 139	+.000 84	−.000 016	+.000 856
1	1	2	−2	−.000 058	0.000 00	.000 00	.000 00	+.000 01	.000 00	+.000 01
−1	−1	2	−2	+.000 844	−0.000 80	.000 00	−.000 80	−.001 40	.000 00	−.001 40
0	−1	2	−2	+.002 027	−0.001 043	−.000 015	−.001 028	−.003 86	−.000 02	−.003 84
1	−1	2	−2	+.000 260	0.000 00	.000 00	.000 00	−.000 15	.000 00	−.000 15
−1	0	0	2	−.000 02	−0.000 05	−.000 02	−.000 03	−.000 02	−.000 04	+.000 01
0	0	0	2	−.000 08	+0.000 5	+.000 287	−.000 23	+.000 16	+.000 56	−.000 40
1	0	0	2	−.000 04	+0.000 6	+.000 11	−.000 05	−.000 07	+.000 23	−.000 30
0	2	0	0	−.000 018	————	————	————	+.000 036	————	+.000 036
0	−2	2	−2	+.000 106	————	————	————	−.000 212	————	−.000 212
0	2	2	−2	−.000 004	————	————	————	+.000 008	————	+.000 008

TABLE XXXVII.

FUNCTIONS OF THE LONGITUDE AND THEIR DERIVATIVES.

Arg.				$\sin \delta v$	$\cos \delta v$	$\frac{1}{2}\sin 2\delta v$,	$\sin^2 \delta v$	$D \cdot \sin^2 \delta v$	$D\delta v$	$\frac{1}{2}D \sin 2\delta v$
g	g'	λ	λ'	sin	cos	sin	cos	cos	sin	sin
0	0	0	0	....	+.996 83	....	+.006 33	+.001 483	....	....
1	0	0	0	+.109 58	−.000 35	+.109 03	+.000 70	+.001 55	.000 000	−.000 320
2	0	0	0	+.003 71	+.003 00	+.003 65	−.005 97	+.000 20	−.000 138	−.000 314
−1	1	0	0	−.000 72	+.000 17	−.000 71	+.000 33	−.000 19	−.002 052	−.002 008
0	1	0	0	−.003 233	−.000 02	−.003 20	+.000 04	+.000 249	−.002 8437	−.002 7992
1	1	0	0	−.000 53	−.000 18	−.000 53	+.000 36	+.000 36	−.001 199	−.001 206
−1	0	2	0	−.000 19	+.000 11	−.000 19	−.000 22	+.000 03	+.000 215	+.000 212
0	0	2	0	−.001 99	−.000 01	−.001 97	+.000 02	−.000 01	+.000 091	+.000 090
1	0	2	0	−.000 22	−.000 11	−.000 22	+.000 22	−.000 01	....	....
−2	0	2	−2	+.001 03	−.001 25	+.001 07	+.002 49	+.005 31	+.002 545	+.002 705
−1	0	2	−2	+.022 17	−.000 58	+.022 03	+.001 16	+.003 77	+.047 405	+.046 777
0	0	2	−2	+.011 45	+.001 17	+.011 35	−.002 34	−.004 83	+.037 838	+.037 341
1	0	2	−2	+.000 96	+.000 67	+ 000 98	−.001 34	−.004 27	+.003 056	+.003 273
2	0	2	−2	+.000 09	+.000 07	+.000 15	−.000 15	−.000 48	+.000 228	+.000 469
−1	1	2	−2	−.000 14	−.000 03	−.000 14	+.000 07	+.000 20	+.000 482	+.000 490
0	1	2	−2	−.000 12	−.000 03	−.000 12	+.000 06	+.000 148	−.000 3608	−.000 3662
1	1	2	−2	−.000 02	....	−.000 02	+.000 01	+.000 08	−.000 055	−.000 075
−1	−1	2	−2	+.001 00	−.000 01	+.000 99	+.000 02	+.000 07	+.002 274	+.002 223
0	−1	2	−2	+.000 80	+.000 07	+.000 80	−.000 16	−.000 451	+.002 9357	+.002 9079
1	−1	2	−2	+.000 07	+.000 05	+.000 08	−.000 10	−.000 39	+.000 271	+.000 305
−1	0	0	2	....	+.000 02	....	−.000 03	−.000 02	....	....
0	0	0	2	−.000 27	+.000 01	−.000 27	−.000 02	−.000 08	−.000 294	−.000 300
1	0	0	2	....	+.000 01	....	−.000 02	−.000 07	....	+.000 008
0	2	0	0	−.000 036	....	−.000 036	....	....	−.000 008	....
0	−2	2	−2	+.000 039	....	+.000 039	....	....	+.000 156	....

TABLE XXXVIII.

COÖRDINATES REFERRED TO MEAN MOON AND THEIR DERIVATIVES.

Arg.				ξ_1 Brown.	$D\xi_1$ Anal.	$\partial\xi_1/\partial e$ Brown.	$\partial\xi_1/\partial\gamma$ Brown.	η_1 Brown.	$D\eta_1$ Anal.	$\partial\eta_1/\partial e$ Brown.	$\partial\eta_1/\partial\gamma$ Brown.
g	g'	λ	λ'	cos	cos	cos	cos	sin	sin	sin	sin
0	0	0	0	+.995 47	−.003 02	−.058 4	−.089 4	....	....	....	....
1	0	0	0	−.054 49	+.001 27	−.995 5	+.004 6	+.109 44	−.000 17	+1.991 4	....
2	0	0	0	+.001 51	+.000 01	+.055 03	+.000 1	+.000 74	+.000 01	+0.026 7	....
−1	1	0	0	−.000 174	+.000 04	−.003 17	....	−.000 643	−.000 38	−0.011 71	....
0	1	0	0	+.000 1034	+.000 33	−.000 346	−.000 063	−.003 2283	−.003 12	−0.002 508	+.001 511
1	1	0	0	+.000 095	−.000 04	+.001 74	....	−.000 440	−.000 38	−0.007 96	....
−1	0	2	0	+.000 04	.000 00	+.000 8	+.001 8	−.000 25	.000 00	−0.004 4	−.010 9
0	0	2	0	+.001 99	+.000 01	....	+.098 1	−.001 99	+.000 10	....	−.097 8
1	0	2	0	+.000 05	.000 00	+.001 0	+.002 5	−.000 05	.000 00	−0.001 0	−.002 5
−2	0	2	−2	−.000 593	−.000 63	−.021 56	+.002 6	+.000 967	+.000 95	+0.035 13	....
−1	0	2	−2	−.010 16	−.019 31	−.184 4	+.001 9	+.022 24	+.044 88	+0.403 7	−.004 7
0	0	2	−2	−.006 36	−.022 85	+.023 7	+.006 8	+.010 32	+.035 21	+0.010 6	−.007 7
1	0	2	−2	+.000 18	+.000 29	+.004 8	....	+.000 20	+.000 40	+0.004 8	....
2	0	2	−2	+.000 02	....	....	....	+.000 02	....	....	....
−1	1	2	−2	+.000 028	+.000 19	+.000 52	....	−.000 118	−.000 39	−0.002 15	....
0	1	2	−2	+.000 0478	+.000 20	−.000 262	−.000 118	−.000 0942	−.000 31	−0.000 169	+.000 132
1	1	2	−2	−.000 002	....	−.000 04	....	−.000 003	....	−0.000 04	....
−1	−1	2	−2	−.000 400	−.000 44	−.007 28	....	+.000 993	+.000 90	+0.018 04	....
0	−1	2	−2	−.000 4506	−.001 55	+.001 256	+.000 337	+.000 7327	+.002 25	+0.000 594	−.000 406
1	−1	2	−2	+.000 013	....	+.000 24	....	+.000 014	....	+0.000 26	....
0	0	0	2	−.000 10	−.000 17	....	−.004 5	+.000 27	+.000 08	....	+.012 2

TABLE XL.

FUNCTIONS OF COÖRDINATES REFERRED TO MEAN SUN.

Arg.				$\xi^2-\eta^2$	$\rho^2-3\zeta^2$	$2\xi\eta$	$D(\xi^2-\eta^2)$	$D(\rho^2-3\zeta^2)$	$2D\xi\eta$
g	g'	λ	λ'	cos	cos	sin	cos	cos	sin
0	0	0	0	−.013 96	+.990 74	....	−.052 095	−.004 447	....
1	0	0	0	−.031 26	−.108 60	−.033 92	−.063 73	+.004 62	−.073 08
2	0	0	0	−.002 50	+.000 96	−.002 43	−.005 55	+.000 07	−.005 47
−1	1	0	0	+.000 13	−.000 71	−.000 08	−.000 58	−.001 76	+.001 37
0	1	0	0	−.000 933	+.000 264	−.001 123	−.003 423	+.000 710	−.004 563
1	1	0	0	−.001 35	+.000 55	−.001 34	−.003 04	+.001 26	−.002 89
−1	0	2	0	−.000 04	−.000 32	+.000 10	−.000 01	−.000 09	−.000 01
0	0	2	0	−.000 38	+.011 96	−.000 38	−.000 42	−.000 15	−.000 42
1	0	2	0	−.000 02	+.000 66	−.000 02	−.000 12	+.000 01	−.000 12
−2	0	2	−2	+.007 809	+.001 807	+.007 185	+.000 30	+.003 73	−.000 60
−1	0	2	−2	−.163 02	−.018 60	−.163 54	+.002 91	−.035 03	−.000 25
0	0	2	−2	+.986 19	−.013 99	+.986 11	−.007 00	−.047 84	−.007 70
1	0	2	−2	+.054 44	−.000 42	+.054 44	+.000 62	−.001 53	+.000 62
2	0	2	−2	+.003 00	−.000 02	+.003 00	−.000 03	−.000 07	−.000 03
−1	1	2	−2	−.000 55	+.000 15	−.000 56	−.002 60	−.000 47	−.002 60
0	1	2	−2	−.003 18	+.000 03	−.003 17	−.002 76	+.000 89	−.002 76
1	1	2	−2	−.000 42	.000 00	−.000 42	−.000 77	+.000 01	−.000 77
−1	−1	2	−2	+.000 27	−.000 87	+.000 28	+.001 33	−.001 40	+.001 33
0	−1	2	−2	+.003 22	−.000 06	+.003 14	+.002 89	−.003 98	+.002 89
1	−1	2	−2	+.000 54	.000 00	+.000 54	+.001 24	−.000 15	+.001 23
1	0	0	2	−.000 20	−.000 32	+.000 24	+.000 03	−.000 76	−.000 03
0	0	0	2	+.003 96	−.000 84	−.003 96	−.000 13	−.001 52	+.000 13
1	0	0	−2	+.000 10	−.000 04	+.000 10	−.000 31	+.000 08	−.000 31
0	2	0	0	−.000 059	+.000 005	−.000 059	−.000 258	+.000 036	−.000 266

Arg.				$\dfrac{\partial(\xi^2-\eta^2)}{\partial e}$	$\dfrac{\partial(\rho^2-3\zeta^2)}{\partial e}$	$2\dfrac{\partial\xi\eta}{\partial e}$	$\dfrac{\partial(\xi^2-\eta^2)}{\partial\gamma}$	$\dfrac{\partial(\rho^2-3\zeta^2)}{\partial\gamma}$	$2\dfrac{\partial\xi\eta}{\partial\gamma}$
g	g'	λ	λ'	cos	cos	sin	cos	cos	sin
0	0	0	0	+0.110 765	+0.170 489	....	+.015 359	−.537 525	....
1	0	0	0	−0.565 7	−1.955 1	−0.608 8	+.009 4	+.050 2	+.009 5
2	0	0	0	−0.073 3	−0.054 5	−0.069 2	....	....	....
−1	1	0	0	−0.000 0	−0.006 4	....	....	....	....
0	1	0	0	+0.004 519	+0.001 109	+0.006 220	+.000 287	−.000 161	+.000 469
1	1	0	0	−0.025 2	+0.003 6	−0.025 6	....	....	....
−1	0	2	0	....	−0.023 0	....	....	−.058 0	....
0	0	2	0	....	−0.003 6	....	−.017 3	+.551 6	−.015 1
1	0	2	0	....	+0.007 8	....	....	+.019 6	....
−2	0	2	−2	+0.284 09	+0.065 69	+0.261 69	....	....	....
−1	0	2	−2	−2.964 1	−0.343 1	−2.964 2	+.018 2	+.005 2	+.018 3
0	0	2	−2	−0.286 3	−0.021 5	−0.286 4	−.178 2	+.041 0	−.178 0
1	0	2	−2	+0.976 9	−0.007 5	+0.976 8	−.006 6	....	−.006 4
2	0	2	−2	+0.109 0	....	+0.109 0	....	....	....
1	1	2	−2	−0.006 4	0.000 0	−0.006 0	....	....	....
−1	−1	2	−2	+0.009 5	−0.014 7	+0.009 6	....	....	....
1	−1	2	−2	+0.008 8	0.000 0	+0.008 4	....	....	....
0	0	0	2	....	....	....	+.195 0	−.035 4	−.195 2

PART IV.

DERIVATION OF RESULTS.

CHAPTER VII.

CONSTANT AND SECULAR TERMS.

We recall the arrangement of the present work. In Part I, the general equations have been formed, the theory outlined, and the methods developed so far as could be done. Nearly all the fundamental quantities were developed as sums of products of two factors, one factor of each pair being a function of the Moon's coördinates, the other a function of the coördinates of the planets. The latter functions are developed in detail in Part II, one chapter of which is devoted to the developments of the coefficients of the direct action, the other to the coefficients of the indirect action. In Part III, Chapter VI, have been developed the numerical functions for the lunar coefficients. These are the same for both actions. The present concluding Part is devoted to the combination of these factors and the derivation and discussion of results.

We may divide the matter of this part into three chapters. In the first chapter we consider the terms not purely periodic. By a purely periodic term is meant one of which the coefficient of the sine or cosine is constant. We may, therefore, define the terms to be first considered as constant and secular, two classes which need not be considered separately.

§ 54. The arguments on which the planetary and lunar factors depend are all distinct except g', which is common to both. It follows that no constant or secular term in the variations of the elements can arise by the multiplication of factors depending on any other variable argument than g'. In all cases in which another argument than this enters into either factor, the results will be periodic in form, the coefficient, however, having, in the general case, a secular variation. Since no terms of the class in question contain l, π, or θ, they give

$$D_t\alpha = 0 \qquad D_t e = 0 \qquad D_t\gamma = 0$$

To form the constant and secular terms we begin by collecting those planetary factors which are either constant or depend on the argument g'. We shall consider the direct and indirect actions separately. The planetary factors for the direct action, as collected from § 42, with some revision of the numbers there found, are shown on the next page.

FACTORS FOR DIRECT ACTION.

Action of Venus.

$$\begin{aligned} 10^3 MK &= +5''.9045 + 0''.44 \cos g' - 0''.11 \sin g' \\ \tfrac{1}{2} 10^3 MC &= -3\ .4072 - 0\ .30 \cos g' + 0\ .07 \sin g' \\ 10^3 MD &= +0''.33 \sin g' + 0''.03 \cos g' \end{aligned}$$

Action of Mars.

$$\begin{aligned} 10^3 MK &= +0''.0468 - 0''.020 \cos g' - 0''.024 \sin g' \\ \tfrac{1}{2} 10^3 MC &= -0\ .1006 + 0\ .028 \cos g' + 0\ .029 \sin g' \\ 10^3 MD &= +0''.010 \sin g' - 0''.008 \cos g' \end{aligned} \qquad (18)$$

Action of Jupiter.

$$\begin{aligned} 10^3 MK &= +0''.0913 - 0''.002 \cos g' - 0''.032 \sin g' \\ \tfrac{1}{2} 10^3 MC &= -2\ .1348 + 0\ .004 \cos g' + 0\ .062 \sin g' \\ 10^3 MD &= +0''.005 \sin g' - 0''.030 \cos g' \end{aligned}$$

Action of Saturn.

$$10^3 MK = +0''.0013 \qquad \tfrac{1}{2} 10^3 MC = -0''.1040 \qquad 10^3 MD = 0$$

The corresponding factors for the indirect action have been combined for the five disturbing planets, Venus to Uranus. From the combined values of G, J, and I, reached in § 44, we find, including Uranus, but omitting Mercury:

$$\begin{aligned} 10^3 m^2 G &= +0''.459 + 0''.36 \cos g' + 0''.06 \sin g' \\ 10^3 m^2 J &= +0\ .153 + 0\ .12 \cos g' + 0\ .02 \sin g' \\ 10^3 m^2 I &= -0\ .03 \sin g' \end{aligned}$$

§ 55. *Lunar Factors.* If, for brevity, we put F for any one of the three lunar factors, say

$$F = \xi^2 - \eta^2 \qquad F' = \rho^2 - 3\zeta^2 \qquad F'' = 2\xi\eta \qquad (19)$$

the terms of the fundamental equations (42) or (57) corresponding to each F will be:

$$\begin{aligned} &\text{in } -D_{nt} l_0; \quad \alpha_1 D' F + e_1 \frac{\partial F}{\partial e} + \gamma_1 \frac{\partial F}{\partial \gamma} \equiv F_1 \\ &\text{in } -D_{nt} \pi_0; \quad \alpha_2 D' F + e_2 \frac{\partial F}{\partial e} + \gamma_2 \frac{\partial F}{\partial \gamma} \equiv F_2 \\ &\text{in } -D_{nt} \theta_0; \quad \alpha_3 D' F + e_3 \frac{\partial F}{\partial e} + \gamma_3 \frac{\partial F}{\partial \gamma} \equiv F_3 \end{aligned}$$

From the tabular values of the functions of the coördinates and their derivatives in Table XL, p. 112, noting that symbolically, $D' = D+2$, we have the following values of the terms of these functions which are independent of the lunar arguments

$$\begin{aligned} F &= \xi^2 - \eta^2 = -.013\ 96 - .000\ 933 \cos g' - .000\ 06 \cos 2g' \\ F' &= \rho^2 - 3\zeta^2 = .990\ 74 + .000\ 264 \cos g' \\ F'' &= 2\xi\eta = -.001\ 123 \sin g' - .000\ 06 \sin 2g' \end{aligned}$$

$$
\begin{aligned}
D'F &= D'(\xi^2 - \eta^2) = -\,.080\,02 - .005\,288 \cos g' - .000\,38 \cos 2g' \\
D'F' &= D'(\rho^2 - 3\zeta^2) = 1.9770 + .001\,239 \cos g' + .000\,04 \cos 2g' \\
D'F'' &= 2D'\xi\eta = -\,.006\,809 \sin g' - .000\,39 \sin 2g'
\end{aligned}
$$

$$\frac{\partial F}{\partial e} = \frac{\partial(\xi^2 - \eta^2)}{\partial e} = +\,.1108 + .004\,519 \cos g'$$

$$\frac{\partial F'}{\partial e} = \frac{\partial(\rho^2 - 3\zeta^2)}{\partial e} = +\,.1705 + .001\,109 \cos g'$$

$$\frac{\partial F''}{\partial e} = \frac{2\partial\xi\eta}{\partial e} = +\,.006\,220 \sin g'$$

$$\frac{\partial F}{\partial \gamma} = \frac{\partial(\xi^2 - \eta^2)}{\partial \gamma} = +\,.0154 + .000\,287 \cos g'$$

$$\frac{\partial F'}{\partial \gamma} = \frac{2(\rho^2 - 3\zeta^2)}{\partial \gamma} = -\,.5375 - .000\,161 \cos g'$$

$$\frac{\partial F''}{\partial \gamma} = \frac{2\partial\xi\eta}{\partial \gamma} = +\,.000\,469 \sin g'$$

The factors a_i, e_i, etc., are derived in § 14, and found in (26). From the preceding scheme we find by using the preceding values and their derivatives in (20)

$$
\begin{aligned}
F_1 &= -\,0.1641 - .010\,92 \cos g' - .000\,77 \cos 2g' \\
F_2 &= -\,2.1194 - .086\,40 \cos g' + .000\,01 \cos 2g' \\
F_3 &= -\,0.0859 - .001\,63 \cos g' \\
F_1' &= +\,4.0086 + .002\,49 \cos g' + .000\,08 \cos 2g' \\
F_2' &= -\,3.3142 - .021\,36 \cos g' \\
F_3' &= +\,3.0093 - .000\,90 \cos g' \\
F_1'' &= -\,.013\,89 \sin g' - .000\,79 \sin 2g' \\
F_2'' &= -\,.118\,94 \sin g' + .000\,01 \sin 2g' \\
F_3'' &= -\,.002\,64 \sin g'
\end{aligned}
\qquad (20)
$$

§ 56. *Secular motions of l, π, and θ.* The function MH, as defined in § 20, may now be written

$$MH = MKF - \tfrac{1}{2}MC_1F' + MD_1F''$$

and introducing the linear functions of its derivatives which we have just formed we have from (42)

$$
\begin{aligned}
D_{nt}l_0 &= -\,MKF_1 + \tfrac{1}{2}MC_1F_1' - MDF_1'' \\
D_{nt}\pi_0 &= -\,MKF_2 + \tfrac{1}{2}MCF_2' - MDF_2'' \\
D_{nt}\theta_0 &= -\,MKF_3 + \tfrac{1}{2}MCF_3' - MDF_3''
\end{aligned}
\qquad (21)
$$

of all which factors we have just given the numerical values. For the indirect action the second members are

$$m^2GF_i + m^2JF_i' - m^2IF_i'' \cdots (i = 1, 2, 3) \qquad (22)$$

Performing the multiplications we find the following secular motions of l_0, π_0, and θ_0 arising from the terms of direct and indirect action under consideration.

Direct Action of	$10^3 D_{nt} l_0$	$10^3 D_{nt} \pi_0$	$10^3 D_{nt} \theta_0$
Venus	− 12.66	+ 23.846	− 9.747
Mars	− 0.41	+ 0.433	− 0.300
Jupiter	− 8.54	+ 7.269	− 6.416
Saturn	− 0.42	+ 0.346	− 0.318
Uranus	− 0.01	+ 0.007	− 0.006
Sum	− 22.04	+ 31.901	− 16.786
Indirect	+ 0.54	− 1.495	+ 0.422
Total	− 21.50	+ 30.406	− 16.364

Taking the Julian century as unit of time $n = 8400$. The centennial motions arising from the factors here employed are therefore:

Centennial motion of $l_0 = -180''.6$, of $\pi_0 = +255.41$, of $\theta_0 = -137.46$. (23)

From the vanishing of $D_t a$, $D_t e$, and $D_t \gamma$ we have

$$\delta n = \text{const.} \qquad \delta\pi_1 = \text{const.} \qquad \delta\theta_1 = \text{const.}$$

the constants being functions of the arbitrary constants of integration, determined at the end of this chapter.

§ 57. *Terms arising from the secular variation of the earth's eccentricity.*

Both the direct and indirect actions contain, in rigor, terms of this class. They enter into the direct action because the direct action of the planet on the Moon varies with the variation of the orbit of the earth around the Sun. But the effect of this variation is found to be so slight that it will be left out of consideration in the present work. We therefore begin with the indirect action. The terms of the coefficients G, I, and J, on which the action depends, have been developed in Chapter V, § 44.

Our fundamental quantity for the indirect action is H' of § 25, of which the only terms required are

$$H' = -G(\xi^2 - \eta^2) - J(\rho^2 - 3\zeta^2) + 2I\xi\eta = -GF - JF' + IF'' \tag{24}$$

The terms of G, I, and J required for the present purpose are

$$G = G_1 \Delta e' \qquad I = I_1 \Delta e' \qquad J = J_1 \Delta e'$$

G_1, I_1, and J_1 being found in § 44 and

$$\Delta e' = -8''.595\,T - 0''.0260\,T^2 = -8''.595\,T(1 + .00302\,T)$$

The secular terms of these coefficients thus become

$$\begin{aligned} G &= (-0''.5358 + 19''.29 \cos g' + 1''.83 \cos 2g')T(1 + .00302\,T) \\ J &= (+0''.1080 + 6''.44 \cos g' + 0''.32 \cos 2g')T(1 + .00302\,T) \\ I &= (-25''.45 \sin g' - 1''.84 \sin 2g')T(1 + .00302\,T) \end{aligned} \tag{25}$$

Using these values in (56) we find

$$D_{nt}a = D_{nt}e = D_{nt}\gamma = 0$$

If we put G', J', and I' for the coefficients of T in (25) we shall have from (20) and (22) the following computation for the secular accelerations from the fundamental equations (57), in which only the non-periodic terms are to be used :

$$\begin{aligned}
[l] &= G'F_1 + J'F_1' - I'F_1'' = +0''.2467 \\
[\pi] &= G'F_2 + J'F_2' - I'F_2'' = -1\ .6378 \\
[\theta] &= G'F_3 + J'F_3' - I'F_3'' = +0\ .3246
\end{aligned}$$

Then, postponing the terms in T^2

$$D_{nt}l = m^2[l]\,T \qquad D_{nt}\pi = m^2[\pi]\,T \qquad D_{nt}\theta = m^2[\theta]\,T \tag{26}$$

The terms in T^2 in (25) are only those arising from the term of e' in T^2. To find the complete values we note that all the terms of $[l]$, $[\pi]$, and $[\theta]$ contain e' as a factor, and may therefore be expressed in the form $e'k$, k being a quantity which, though containing minute terms in e'^2, may be regarded as a constant. Then

$$D_t[l] = kD_te' = [l]\frac{D_te'}{e'} = -.002495[l]$$

and the actual values of $[l]$, $[\pi]$, and $[\theta]$ may be written in the form

$$[l] = [l]_0(1 - .00250T)(1 + .00302T) = [l]_0(1 + .00052T)$$

$[l_0]$, etc., being the values computed above. Multiplying by T we find that the terms of D_tl, $D_t\pi$, and $D_t\theta$ in T^2 are found from those in T by multiplying the latter by the factor $+.00052T$.

Taking the Julian century as the unit of time, $m^2n = 46.998$, whence

$$D_tl = +11''.60T + 0''.0060T^2 \quad D_t\pi = -76''.98T - 0''.040T^2 \quad D_t\theta = +15.25T + 0.0079T^2$$

Then by integration

$$\delta l = 5''.80T^2 + 0''.0020T^3 \quad \delta\pi = -38''.49T^2 - 0''.013T^3 \quad \delta\theta = 7''.62T^2 + 0''.0026T^2 \tag{27}$$

This value of the secular acceleration of the mean longitude is, I believe, markedly smaller than any heretofore found. Delaunay's last result was 6''.11, which, reduced to the now adopted value of the secular diminution of e', would become 6''.02. The necessity of using Delaunay's development of the parallax in forming the D's of some of the coefficients leads to some uncertainty in the present result. But my rough estimate would lead to the conclusion that the uncertainty should be less than one per cent. of the whole amount. The question of the precision of the value here reached I must leave to other investigators.[*]

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[*] As this work is going through the press the author notices that Brown's value found in *Monthly Notices Royal Astronomical Society*, vol. LVII, is reduced from 5''.91 to 5''.81 when the now adopted D_te' is used.

§ 58. We have next to consider the secular variations of the periodic terms in general. Taking any set of such terms depending on any argument N

$$\zeta^2 - \eta^2 = 2p \cos N \qquad \rho^2 - 3\zeta^2 = 2q \cos N \qquad 2\xi\eta = \kappa_4 \sin N$$

we shall have the terms of H' in (24)

$$\begin{aligned} H' &= (1''.072p - 0''.216q)T\cos N + (-38''.6p - 12''.9q)T\cos g' \cos N - 25''.4\kappa_4 T \sin g' \sin N \\ &= (1''.072p - 0''.216q)T\cos N - (19''.3p + 6''.4q + 12''.7\kappa_4)T\cos(N-g') \\ &\qquad - (19''.3p + 6''.4q - 12''.7\kappa_4)T\cos(N+g') \end{aligned}$$

Forming the partial derivatives of these terms of H' as to l, π, θ, α, e, and γ, and carrying them into the fundamental equations (64) and (65) by the processes of § § 22 and 23 we shall be led to

$$\begin{aligned} D_{nt}a = m^2(-1''.072\mathrm{a}p + 0''.216\mathrm{a}q)T\sin N + m^2(19''.3\mathrm{a}p + 6''.4\mathrm{a}q + 12''.7\mathrm{a}\kappa_4)T\sin(N-g') \\ + m^2(19''.3\mathrm{a}p + 6.4\mathrm{a}q - 12''.7\mathrm{a}\kappa_4)T\sin(N+g') \end{aligned} \tag{28}$$

with similar equations for e and γ formed by writing e and g respectively for a. Also, we shall have

$$\begin{aligned} D_{nt}l_0 = m^2(1''.072L' - 0''.216L'')T\cos N^2 - m^2(19''.3L' + 6''.4L'' + 12''.7L_4)T\cos(N-g') \\ - m^2(19''.3L' + 6''.4L'' - 12''.7L_4)T\cos(N+g') \end{aligned} \tag{29}$$

with similar equations for $D_{nt}\,\pi_0$ and $D_{nt}\,\theta_0$ formed by writing P and R respectively for L

§ 59. The special values of N of most importance in the present connection are

$$N = 0 \qquad N = g' \qquad N = g$$

on which depend, respectively, the constant term, the annual equation, the equation of the center, and the evection

CASE I; $N=0$.

The factors for $D_{nt}a$, $D_{nt}e$, and $D_{nt}\gamma$ all vanish. The values of the L-coefficients are found in the first line of Table XLIX, p. 147. The first or purely secular term of (29) has already been computed. The remaining terms give

$$\begin{aligned} D_t l_0 &= m^2n(38''.6L' + 12''.9L'')T\cos g' \\ D_t\pi_0 &= m^2n(38\ .6P' + 12\ .9P'')T\cos g' \\ D_t\theta_0 &= m^2n(38\ .6R' + 12\ .9R'')T\cos g' \end{aligned}$$

Substituting the numerical values of L, P, R, and $m^2n = 47.00$;

$$D_t l_0 = +1062''T\cos g' \qquad D_t\pi_0 = -2961''T\cos g' \qquad D_t\theta_0 = +827''T\cos g'$$

We cite, for convenient reference, the following indefinite integrals

$$\int t \sin \mathrm{N}t\,dt = \frac{1}{\mathrm{N}^2}\sin \mathrm{N}t - \frac{t}{\mathrm{N}}\cos \mathrm{N}t \qquad \int t \cos \mathrm{N}t\,dt = \frac{1}{\mathrm{N}^2}\cos \mathrm{N}t + \frac{t}{\mathrm{N}}\sin \mathrm{N}t$$

The unit of t in these equations being 100 years, N is the motion of g' in this period, for which we may take 200 π, or N $=628$.

Integration by the above formulæ then gives

$$\begin{aligned} \delta l_0 &= +1''.69\,T\sin g' + 0''.003\cos g' \\ \delta\pi_0 &= -4.71\,T\sin g' - .008\cos g' \\ \delta\theta_0 &= +1.32\,T\sin g' + .002\cos g' \end{aligned} \tag{30}$$

CASE II; $N=g'$; *the annual term.*

Here also the variations of a, e, and γ vanish, so that only those of l_0, π_0, and θ_0 are affected. Carrying into the equations (29) the numerical values of the lunar coefficients for the Arg. g' we find, dropping the constant terms, which have been already computed,

$$\begin{aligned} D_t l_0 &= m^2 n T(0''.0061\cos g' - 0''.082\cos 2g') = 0''.29\,T\cos g' - 3''.80\,T\cos 2g' \\ D_t\pi_0 &= m^2 n T(0.0485\cos g' + 0.131\cos 2g') = 2.28\,T\cos g' + 3.44\,T\cos 2g' \\ D_t\theta_0 &= m^2 n T(0''.0010\cos g' + 0''.022\cos 2g) = 0''.47\,T\cos g' + 1.03\,T\cos 2g' \end{aligned}$$

Then, integrating, and dropping insignificant constant coefficients

$$\begin{aligned} \delta l_0 &= +0''.00047\,T\sin g' - 0''.0036\,T\sin 2g' \\ \delta\pi_0 &= +0.0037\,T\sin g' - 0.0028\,T\sin 2g' \\ \delta\theta_0 &= +0.0008\,T\sin g' - 0.0016\,T\sin 2g' \end{aligned}$$

CASE III; $N=g$.

For this argument I have used the following preliminary values of the lunar coefficients, differing from those of Tables XLVIII and XLIX by amounts here unimportant

$$\begin{array}{lll} L' = -0.1197 & L'' = -0.1962 & L_4 = -0.2648 \\ P' = +5.403 & P'' = +18.734 & P_4 = +11.688 \\ R' = -0.0028 & R'' = -0.1431 & R_4 = -0.0560 \\ \mathrm{a}p = -0.032\,23 & \mathrm{a}q = -0.110\,49 & \mathrm{a}\kappa_4 = -0.069\,70 \\ \mathrm{e}p = -0.300\,45 & \mathrm{e}q = -1.033\,40 & \mathrm{e}\kappa_4 = -0.650\,66 \\ \mathrm{g}p = +0.000\,09 & \mathrm{g}q = +0.000\,31 & \mathrm{g}\kappa_4 = +0.000\,10 \end{array}$$

Carrying these values into the equations (28) and (29) we find, for the terms depending on the argument g alone,

$$\begin{aligned} D_t a &= +0''.520\,T\sin g & \qquad D_t e &= +4''.70\,T\sin g \\ D_t l_0 &= +4''.04\,T\cos g & \qquad D_t\pi_0 &= -82''.2\,T\cos g \end{aligned}$$

For the motion of g, N $=8329$.
Integration then gives

$$\begin{aligned} \delta a &= -0''.000\,062\,T\cos g + (74'' + 10^{10})\sin g & \qquad \delta e &= -0''.0056\,T\cos g \\ \delta l_0 &= +0''.000\,485\,T\sin g & \qquad \delta\pi_0 &= -0''.009\,87\,T\sin g \end{aligned}$$

We drop the terms with constant coefficients, owing to their minuteness, and find, with $n = 8400$;

$$\delta n = -\tfrac{3}{2} n \delta a = +0''.78\, T \cos g$$

Then by integration

$$\delta_n l = \int \delta n dt = +0''.000\,094\, T \sin g$$

This, added to δl_0, gives for the entire term in δl

$$\delta l = +0''.000\,578\, T \sin g \qquad (31)$$

§ 60. In order to determine the complete expressions for the coördinates themselves, the terms computed in the present section, together with those which may be found in a similar way for the other periodic terms, are to be carried into the expression for the Moon's true longitude in terms of the elements. I have not, however, deemed it necessary to do this in the case of the secular variations of the periodic terms, because these can be most readily determined by varying the value of e' in the Delaunay or Brown expressions for the Moon's longitude.

I have, however, computed the preceding variations of some terms owing to the theoretical interest which attaches to the relations implied by the equality of the result of the present method to those of the other method. The two methods correspond to the two methods by which the secular acceleration may be determined. In *Action*, p. 191, it is shown that the secular acceleration of l, π, and θ may be derived from the secular change of e' by determining the corresponding secular changes in a, e, and γ. This theorem has been discussed and extended by Brown in his paper on *Transmitted Motions and Indirect Perturbations.*[*]

By this method the secular variations in question appear as variations of n, π_1, and θ_1, the latter being functions of the variables a, e, and γ. But, in the present theory, a, e, and γ remain constant so far as the secular change of e' is concerned, and the changes are thrown wholly upon l_0, π_0, and θ_0.

There is therefore a seeming contradiction in that the lunar elements a, e, and γ are affected by a secular variation in one theory, while in the other they are practically constant. Referring to Brown's paper for the theory of the subject it will be instructive to show the relation between the two methods.

In what I have, for brevity, called the Delaunay solution of the problem, the Moon's coördinates appear as functions of the lunar elements, introduced as arbitrary constants, and of the Sun's eccentricity, which is regarded as a quantity given in advance. But, when the action of the planets is introduced, the solar element e', as well as the lunar elements a, e, and γ, become variable. In what I may call method A of treating the planetary action, which was that adopted in *Action*, the final values of the coördinates as affected by planetary action are determined by introducing the simultaneous variations of all four elements into the Delaunay expressions. But in method *B*, adopted in the present work, the entire variations have been thrown upon the lunar elements, the solar elements being regarded as constant. In the case of the periodic perturbations this course is practically a necessity, owing to the extreme complexity introduced into the formulæ if we suppose the coördinates expressed in terms of the value of e' affected by periodic inequalities. But it is different in the case of the secular motion of e'. Here it is more logical to consider that at any epoch the action of the Sun is computed with the actual eccentricity at that epoch, and so to use method *A*.

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[*] *Transactions of the American Mathematical Society*, vol. VI, p. 332. See also, *Monthly Notices, Royal Astronomical Society*, vol. LVII.

Not having done this in the present work, but having regarded the value of e' at the epoch 1850 as a fundamental constant, the values of G, J, and I, though functions of e', and therefore variable, have appeared in the theory as constants.

In the present investigation the author has not, for want of time, investigated the modifications which would be made in the problem if these coefficients were taken as affected by their secular variations. One reason for refraining from this course was that the determination of the secular acceleration from the equations given in *Action*, page 191, require a much more extended development of the canonical elements in terms of e' than it was practicable to undertake in the present paper. The question is therefore left to others, reference being made to Brown's paper on the variation of given and arbitrary constants.*

A comparison of the secular variation of the coefficient of sin g' with that found by Delaunay's value of this term will, however, be of interest. With the eccentricity of 1850 the coefficient of this annual term is $-670''$. It contains e' as a factor, the portion arising from higher powers of this element being unimportant in the present case. It follows that the secular variation of the coefficient of sin g' in δv is

$$-670''\frac{\delta e'}{e'} = +1''.67\,T$$

The term found in (30) for δl is $1''.69\ T$. I have not computed δv itself.

The two methods of treating the effect of the motion of the ecliptic are related to each other in the same way as this just discussed. Had the method of the present paper been strictly followed throughout, the coördinates of the Moon would have been referred to a fixed ecliptic, because the ecliptic remains fixed when planetary action is omitted. But it was seen that by a very slight and easily determined change, the coördinates could be referred to the actually moving ecliptic, and and the work was carried on accordingly. In concluding the work, it is a matter of regret to the author that he did not investigate the question whether the Moon's coördinates could not, on the same principle, be expressed in terms of a varying solar eccentricity, *ab initio*, thus simplifying the problem in conception at least. Owing, however, to the theoretical interest attaching to the relation between the two methods, the effect of the motion of the ecliptic might be treated by both methods.

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* *L. c.*, vol. IV, p. 333.

§ 61. *Adjustment of the Arbitrary Constants.* The problem before us may be outlined thus. The preliminary solution of the problem of three bodies leads to expression of the Moon's coördinates as functions of six arbitrary constants, through the intermediary of three other functions of these constants l, π, and θ. The solution in terms of the six elements a, e, γ, l, π, θ takes the form:

$$l = l_0 + nt \qquad \pi = \pi_0 + \pi_1 t \qquad \theta = \theta_0 + \theta_1 t \qquad v = l + \phi \qquad \rho = 1 + \phi \qquad \beta = \phi$$

the functions ϕ being of a form not necessary to specify at present. As already mentioned, n, π_1, and θ_1 are functions of a (or n), e, and γ. The solution of our problem is now completed by adding to the expressions for the Moon's coördinates $\equiv v$, r, and β, the quantities

$$\delta v = \frac{\partial v}{\partial \alpha}\delta\alpha + \frac{\partial v}{\partial l}\delta l + \frac{\partial v}{\partial e}\delta e + \frac{\partial v}{\partial \pi}\delta\pi + \cdots \tag{32}$$

with similar forms for r and β, which we need not write. For our present purpose it will be necessary and sufficient to consider the following terms in v, the true longitude.

$$v = l + 2e\sin(l - \pi)$$

We then have

$$\frac{\partial v}{\partial l} = \frac{\partial v}{\partial l_0} = 1 + 2e\cos g \qquad \frac{\partial v}{\partial \pi} = -2e\cos g$$

$$\frac{\partial v}{\partial e} = 2\sin g + \frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial e} \qquad \frac{\partial v}{\partial \alpha} = \frac{\partial v}{\partial l}\frac{\partial l}{\partial \alpha} + \frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial \alpha}$$

Substituting in (32) and omitting unimportant terms

$$\delta v = -\tfrac{3}{2}nt(1 + 2e\cos g)\delta\alpha - 2e\cos g\,\delta\pi + (1 + 2e\cos g)\,\delta l_0 + 2\sin g\,\delta e$$

We put

$$\delta\alpha_0,\ \delta e_0,\ \delta\pi_0,\ \delta l_0,$$

the arbitrary constants to be added to the perturbations $\delta\alpha$, δe, $\delta\pi$, and δl. We then have the following perturbations depending on the purely lunar arguments

$$\delta\alpha = \delta\alpha_0 - 0''.0016\sin g \qquad \delta e = \delta e_0 - 0''.0150\cos g$$
$$\delta l = \delta l_0 - {}''.0212nt + {}''.0059\sin g \qquad \delta\pi = \delta\pi_0 + {}''.0315nt - {}''.272\sin g$$

Substituting in the derivatives we have the result that the mean sidereal motion of the Moon is

$$nt(1 - 0''.0212 - \tfrac{3}{2}\delta_0\alpha)$$

We now determine $\delta_0\alpha$ by the condition that the mean motion shall be represented by n. Thus

$$\delta_0\alpha = -0''.0141 \qquad \delta_0 n = 0''.0212n = +178'' \tag{33}$$

Also, the coefficient of sin g in the expression for the longitude becomes

$$2e + 0''.0059 + 2\delta_0 e$$

We now determine $\delta_0 e$ by the condition that the expression for the coefficient shall remain unchanged. This gives

$$\delta_0 e = -0''.003$$

The longitudes, perigee, and node being given by the equations

$$\pi = \pi_0 + \pi_1 t \qquad \theta = \theta_0 + \theta_1 t$$

the introduction of the perturbations of the elements will give rise to the increments

$$\delta\pi_1 = \frac{\partial\pi_1}{\partial n}\delta n + \frac{\partial\pi_1}{\partial e}\delta e + \frac{\partial\pi_1}{\partial\gamma}\delta\gamma \qquad \delta\theta_1 = \frac{\partial\theta_1}{\partial n}\delta n + \frac{\partial\theta_1}{\partial e}\delta e + \frac{\partial\theta_1}{\partial\gamma}\partial\gamma$$

The effects $\delta_0 e$ and $\delta_0\gamma$ are inappreciable. Taking only $\delta_0 n$ from (33) we have

$$\delta\pi_1 = -.014\,80\delta_0 n = -0''.000\,314n \qquad \delta\theta_1 = -.001\,01\delta_0 n = -''.000\,021n$$

Taking the century as the unit, the adjustment gives

$$\delta\pi_1 = -2''.64 \qquad \delta\theta_1 = -0''.18 \qquad \text{and} \qquad \delta\pi = -2''.64T \qquad \delta\theta = -0''.18T$$

Adding thereto the secular terms of π_0 and θ_0 already found, we have the following results, tor the entire secular effect of the action of the planets on π and θ

	$D\pi$	$D\theta$
Direct action of the planets Venus to Uranus	$+267''.97$	$-141''.00$
Indirect action of the planets Venus to Uranus	$-12\ .56$	$+3\ .54$
Total action of Mercury ($m = 10^{-7}$)	$+0\ .45$	$-0\ .21$
Adjustment of elements	$-2\ .64$	$-0\ .18$
Sum	$+253\ .22$	$-137\ .85$

This motion of the perigee, greater by 5″ than that found by Brown, goes to confirm his conclusion that the gravitation of the Earth does not deviate from Newton's law of the inverse square.

§ 62. As the reason for the last correction may not be quite clear, it may be of interest to state in a general way how it enters into the theory. The action of the planets on the Moon is found on the supposition of what we may call an undisturbed orbit of the Moon, meaning thereby an orbit in which the action of the Sun is completely taken account of, on the supposition that no other extraneous action enters. We thus have a certain mean motion n determined from observations, and a certain undisturbed mean distance, a, determined by the relation $a^3n^2 = \mu$, which requires a constant Δa of correction to the mean distance computed from the action of the Sun, giving rise to an expression for the constant of the Moon's radius vector $a + \Delta a_1 \equiv a_1$ completely representing the action of the Sun on the supposition of no planetary action.

It is with this mean distance a_1 that the actions of the planets, both direct and indirect, are computed. But, as a matter of fact, the action of the planet modifies the relation between a_1 and n, so that we must change either the mean motion or the mean distance according to what values of the elements we assume. If we take the arbitrary constants so that the mean motion remains unchanged, then the actual mean distance will require a constant correction on account of the action of the planets. If we regard the mean distance as an invariable quantity, then there will be a correction to the mean motion.

It follows by either method that when we compute the motion of the perigee and node under the action of the Sun alone, we must make one or the other of these modifications produced by the action of the planet, and determine the effect upon the motion of π and θ. If we regard the actually observed mean motion as that due to the Sun alone then we must introduce a correction to the mean distance, and determine its effect upon π_1 and θ_1. But if, which is the more natural method, we regard the mean distance of the Moon as the given actual element, then we must compute that part of the motion of the perigee and node due to the Sun alone with a different n from that given by observation; that is, with a value found by subducting the planetary effect from the observed value.

We may therefore regard the corrections $-2''.64$ and $-0''.18$ to π_1 and θ_1 as reducing π_1 and θ_1 to their true values under the action of the Sun alone.

§ 63. *Secular Variation of e.* If we require, as we should, that the coefficient of sin g in the Moon's true longitude should be represented by a function of e then the expression (31) shows that this element will be affected by the secular variation

$$\delta e = +0''.00029\,T$$

This being less than $0''.01$ in a thousand years, is of no practical importance, though of theoretical interest.

It may also be remarked in the present connection that the existence of this variation, and the approximate algebraic expression for its amount, was first made known by Adams.*

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* *Monthly Notices, Royal Astronomical Society*, vol. XIX, p. 207.

CHAPTER VIII.

SPECIAL PERIODIC INEQUALITIES.

§ 64. *Reduction to the moving ecliptic.* Since when the Sun is the disturbing body the plane of the ecliptic remains fixed, the inequalities of the coördinates so far reached are referred to the ecliptic of any date regarded as fixed. The only way in which they are affected by the motion of the ecliptic is through the secular variations of the coördinates of the planet arising from that motion. The effects of these are supposed to be too small to need consideration at present. It is, however, necessary to refer the elements to the moving ecliptic. I have shown in § 4 how this may be done by the simple device of adding to the perturbative function the terms

$$\Delta R = 2z(pD_tx_1 - qD_ty_1) + 2(qy - px)D_tz_1 \tag{33}$$

and then integrating the portions of the differential equations thus arising. In this expression p and q are the coefficients expressing the speed of rotation of the ecliptic around the axes of y and x respectively, and are found by putting

Π, the longitude of the ascending node of the moving on the fixed ecliptic;

κ, the speed of rotation.

Then

$$p = \kappa \sin \Pi \qquad q = \kappa \cos \Pi \tag{34}$$

It is to be noted that κ is here used as the speed of rotation, and not as the actual angle rotated through. It is, therefore, of dimension T^{-1} and the expression for ΔR is of dimensions $L^2 T^{-2}$, which, by introducing the dimensions of mass, become identical with the dimensions of P as hitherto used.

The partial derivatives of ΔR as to the lunar elements are to be taken only as they enter through x, y, and z, so that the D_t of the Moon's coördinates, the latter being called for this purpose x_1, y_1, and z_1, are to be regarded as numerically given quantities.

To form the partial derivatives of x, y, and z we use the developments of these coördinates in terms of the lunar elements already given, substituting in x, y, and z the values of ξ, η, and ζ. But in this part of the work it will be convenient to refer the coördinates x and y to a general fixed X-axis, instead of the mean Sun, as heretofore. When this is done the expressions for the ratios of the coördinates to a take the form

$$\xi = \Sigma k \cos N \qquad \eta = \Sigma k \sin N \qquad \zeta = \Sigma c \sin N' \tag{35}$$

where N and N' are of the general form

$$N = il + i_1\pi + i_2\theta + jl' + j_1\pi' \qquad N' = i'l + i_1'\pi + i_2'\theta + j'l' + j_1'\pi'$$

the indices satisfying the conditions

$$i + i_1 + i_2 + j + j_1 = 0 \qquad i' + i_1' + i_2' + j' + j_1' = 1$$

In forming the D_t's of these expressions we put n, n', the *ratios* of the motion of the arguments N or N' to n, that of the Moon. We then have

$$D_t x_1 = -an\Sigma k\mathrm{n} \sin N \qquad D_t y_1 = an\Sigma k\mathrm{n} \cos N \qquad D_t z_1 = an\Sigma c\mathrm{n}' \cos N'$$

The values (34) of p and q then give

$$pD_t x_1 - qD_t y_1 = -an\kappa\Sigma k\mathrm{n} \cos (N - \Pi) \tag{36}$$

$$qy - px = a\kappa\Sigma k \sin (N - \Pi) \tag{37}$$

Our next step is to form the derivatives of z and $qy - px$ as to the lunar elements. The partial derivatives as to z are found from the last equation (35)

$$Dz = a\Sigma D'c \sin N' \qquad \frac{\partial z}{\partial e} = a\Sigma \frac{\partial c}{\partial e} \sin N' \qquad \frac{\partial z}{\partial \gamma} = a\Sigma \frac{\partial c}{\partial \gamma} \sin N' \tag{38}$$

$$\frac{\partial z}{\partial l} = a\Sigma i'c \cos N' \qquad \frac{\partial z}{\partial \pi} = a\Sigma i_1'c \cos N' \qquad \frac{\partial z}{\partial \theta} = a\Sigma i_2'c \cos N' \tag{39}$$

By differentiating (37) on the same system we have

$$\left.\begin{aligned} D(qy - px) &= a\kappa\Sigma D'k \sin (N - \Pi) \\ \frac{\partial(qy - px)}{\partial e} &= a\kappa\Sigma \frac{\partial k}{\partial e} \sin (N - \Pi) \\ \frac{\partial(qy - px)}{\partial \gamma} &= a\kappa\Sigma \frac{\partial k}{\partial \gamma} \sin (N - \Pi) \end{aligned}\right\} \tag{40}$$

$$\left.\begin{aligned} \frac{\partial(qy - px)}{\partial l} &= a\kappa\Sigma ik \cos (N - \Pi) \\ \frac{\partial(qy - px)}{\partial \pi} &= a\kappa\Sigma i_1 k \cos (N - \Pi) \\ \frac{\partial(qy - px)}{\partial \theta} &= a\kappa\Sigma i_2 k \cos (N - \Pi) \end{aligned}\right\} \tag{41}$$

We next have to form the products of (36) by the derivatives (38) and (39) and of $D_t z_1$ by (40) and (41), and form their several sums. We thus find that the combination of any term of argument N with any term of argument N' gives rise to the following terms in the partial derivatives as to e and l:

$$\frac{\partial \Delta R}{\partial e} = a^2 n\kappa \left\{ \mathrm{n}'c\frac{\partial k}{\partial e} - \mathrm{n}k\frac{\partial c}{\partial e} \right\} \sin(N + N' - \Pi) + a^2 n\kappa \left\{ \mathrm{n}'c\frac{\partial k}{\partial e} + \mathrm{n}k\frac{\partial c}{\partial e} \right\} \sin(N - N' - \Pi)$$

$$\frac{\partial \Delta R}{\partial l} = a^2 n\kappa ck \{i\mathrm{n}' - i'\mathrm{n}\} \{\cos(N + N' - \Pi) + \cos(N - N' - \Pi)\} \tag{42}$$

The derivatives as to log a and γ are formed from the first of these equations by simple substitution. Those as to π and θ are formed from the last equation by writing i_1 and i_2 for i, and i'_1 and i'_2 for i'.

The derivatives thus formed being substituted in the fundamental equations the integration of the latter will give the inequalities of the elements. It will be convenient to use the following formulæ of substitution. We first put, in the combination of any term of argument N with any term of argument N':

$$\begin{aligned} k_a &= \mathrm{n}'cD'k + nkD'c & k_a' &= \mathrm{n}'cD'k - nkD'c \\ k_e &= \mathrm{n}'c\frac{\partial k}{\partial e} + \mathrm{n}k\frac{\partial c}{\partial e} & k_e' &= \mathrm{n}'c\frac{\partial k}{\partial e} - \mathrm{n}k\frac{\partial c}{\partial e} \\ k_\gamma &= \mathrm{n}'c\frac{\partial k}{\partial \gamma} + \mathrm{n}k\frac{\partial c}{\partial \gamma} & k_\gamma' &= \mathrm{n}'c\frac{\partial k}{\partial \gamma} - \mathrm{n}k\frac{\partial c}{\partial \gamma} \end{aligned} \tag{43}$$

$$k_l = (i\mathrm{n}' - i'\mathrm{n})ck \qquad k_\pi = (i_1\mathrm{n}' - i_1'\mathrm{n})ck \qquad k_\theta = (i_2\mathrm{n}' - i_2'\mathrm{n})ck$$

The quantities k_a, k'_a, etc., will then be the coefficients of the constant factor $a^2 n\kappa$ in the expressions for the derivatives of the elements. Substituting ΔR for P_1 in the differential equations (27), p. 18, the latter will reduce to the form

$$\begin{aligned} D_t a &= (\alpha_1 k_l + \alpha_2 k_\pi + \alpha_3 k_\theta)\kappa \{\cos(N + N' - \Pi) + \cos(N - N' - \Pi)\} \\ D_t e &= (e_1 k_l + e_2 k_\pi + e_3 k_\theta)\kappa \{\cos(N + N' - \Pi) + \cos(N - N' - \Pi)\} \\ D_t \gamma &= (\gamma_1 k_l + \gamma_2 k_\pi + \gamma_3 k_\theta)\kappa \{\cos(N + N' - \Pi) + \cos(N - N' - \Pi)\} \\ -D_t l_0 &= (\alpha_1 k_a + e_1 k_e + \gamma_1 k_\gamma)\kappa \sin(N - N' - \Pi) + (\alpha_1 k_a' + e_1 k_e' + \gamma_1 k_\gamma')\kappa \sin(N + N' - \Pi) \\ -D_t \pi_0 &= (\alpha_2 k_a + e_2 k_e + \gamma_2 k_\gamma)\kappa \sin(N - N' - \Pi) + (\alpha_2 k_a' + e_2 k_e' + \gamma_2 k_\gamma')\kappa \sin(N + N' - \Pi) \\ -D_t \theta_0 &= (\alpha_3 k_a + e_3 k_e + \gamma_3 k_\gamma)\kappa \sin(N - N' - \Pi) + (\alpha_3 k_a' + e_3 k_e' + \gamma_3 k_\gamma')\kappa \sin(N + N' - \Pi) \end{aligned} \tag{44}$$

By integrating these equations we shall have, in the case of each argument, a divisor which we may call N, equal to the motion of the argument in the unit of time. The quotient $\kappa \div \mathrm{N}$ expresses the angular motion of the ecliptic during the time required for the argument to move through the unit radian.

In the above differential equations we substitute for α_i, e_i, and γ_i, their numerical values and write, for brevity,

$$\begin{aligned} C_l &= 2.023k_a - 0.017k_e - 0.0229k_\gamma \\ C_\pi &= -0.0301k_a - 19.153k_e - 0.020k_\gamma \\ C_\theta &= 0.0075k_a + 0.0026k_e - 5.570k_\gamma \end{aligned} \tag{45}$$

with similar expressions for the accented quantities, and

$$\begin{aligned} C_a &= 2.023k_l - 0.0301k_\pi + 0.0075k_\theta \\ C_e &= -0.0168k_l - 19.153k_\pi + 0.0026k_\theta \\ C_\gamma &= -0.0229k_l - 0.0200k_\pi - 5.570k_\theta \end{aligned} \tag{46}$$

We also put for brevity

$$A = N - N' - \Pi \qquad A' = N + N' - \Pi$$

The values of N and N′, the coefficients of the time in A and A' respectively take the form

$$\mathrm{N} = (\mathrm{n} - \mathrm{n}')n \qquad \mathrm{N}' = (\mathrm{n} + \mathrm{n}')n$$

and the differential variations become

$$\begin{aligned} D_t a &= C_a \kappa(\cos A + \cos A') & -D_t l_0 &= C_l \kappa \sin A + C_l' \kappa \sin A' \\ D_t e &= C_e \kappa(\cos A + \cos A') & -D_t \pi_0 &= C_\pi \kappa \sin A + C_\pi' \kappa \sin A' \\ D_t \gamma &= C_\gamma \kappa(\cos A + \cos A') & -D_t \theta_0 &= C_\theta \kappa \sin A + C_\theta' \kappa \sin A' \end{aligned} \tag{47}$$

We shall then have by integration

$$\begin{aligned} \delta a &= \frac{\kappa}{\mathrm{N}} C_a \sin A + \frac{\kappa}{\mathrm{N}'} C_a \sin A' & \delta l_0 &= \frac{\kappa}{\mathrm{N}} C_l \cos A + \frac{\kappa}{\mathrm{N}'} C_l' \cos A' \\ \delta e &= \frac{\kappa}{\mathrm{N}} C_e \sin A + \frac{\kappa}{\mathrm{N}'} C_e \sin A' & \delta \pi_0 &= \frac{\kappa}{\mathrm{N}} C_\pi \cos A + \frac{\kappa}{\mathrm{N}'} C_\pi' \cos A' \\ \delta \gamma &= \frac{\kappa}{\mathrm{N}} C_\gamma \sin A + \frac{\kappa}{\mathrm{N}'} C_\gamma \sin A' & \delta \theta_0 &= \frac{\kappa}{\mathrm{N}} C_\theta \cos A + \frac{\kappa}{\mathrm{N}'} C_\theta' \cos A' \end{aligned} \tag{48}$$

The largest terms which enter into the theory are shown in Table XLI*a*, for Arg. N, and Table XLI*b* for Arg. N'. The coefficients of the principal terms of each have been derived from the numbers given in Part III.

TABLE XLI*a*.

COEFFICIENTS FOR FORMING $px-qy$; ARG. N.

No.	l i	π i_1	θ i_2	l' j_1	π' j_2	n	k	$D'k$	$\partial k/\partial e$	$\partial k/\partial \gamma$	nk
1	1	0	0	0	0	+1.000 00	+.9955	+.9925	−0.0588	+.9955	+.995 5
2	2	−1	0	0	0	+1.991 55	+.0275	+.0275	+0.4980	.0000	+.054 7
3	0	1	0	0	0	+0.008 45	−.0820	−.0815	−1.4934	.0000	−.000 69
4	−1	2	0	0	0	−0.983 1	+.0004	+.0004	+0.0142	.0000	−.000 4
5	2	−1	0	−1	1	+1.916 8	+.0002	+.0001	+0.0038	.0000	+.000 3
6	0	1	0	1	−1	+0.083 2	−.0004	.0000	−0.0070	.0000	.000 0
7	−1	0	2	0	0	−1.008 0	+.0020	+.0020	+0.0002	+.0980	−.002 0
8	−1	0	0	2	0	−0.850 4	−.0083	−.0081	+0.0052	+.0072	+.007 1

TABLE XLI*b*.

COEFFICIENTS FOR ζ; ARG. N'.

No.	l π θ l' π' i' i_1' i_2' j_1' j_2'	n′	c	$D'c$	$\partial c/\partial e$	$\partial c/\partial \gamma$	n′c
1	0 1 −1 0 0	0.012 5	−.007 2	−.0068	−.131	−0.165	−.0001
2	1 0 −1 0 0	1.004 02	+.089 5	+.0893	−.0050	+1.989	+.0899
3	2 −1 −1 0 0	1.995 56	+.002 47	+.0025	+.045	+0.055	+.0050
4	1 0 1 −2 0	0.846 4	+.003 3	+.0098	.000	+0.068	+.0028

It should be added that the coefficients of the smaller terms show only the order of magnitude in each case, and not the precise numerical value. The latter will be required only in the case of terms found to be sensible.

The theorem that all the inequalities have, as a coefficient, the motion of the ecliptic during nearly one-sixth the period of the argument will enable us to limit the combinations of the arguments to be considered.

In the case of any argument $N \pm N'$ containing the Moon's mean longitude, one-sixth the period will never materially exceed 5 days, for which we have

$$\frac{\kappa}{\mathrm{N}} = 0''.0064$$

In none of the terms of this class is there a factor C so large as to bring the coefficient up to 0.″05. It follows that all the combinations $N \pm N'$ which contain the Moon's mean longitude may be omitted.

Of the terms which remain none can have a period several times greater than that of the node, for which the ratio $\kappa : \mathrm{N} = 1''.5$. It follows that no combinations of arguments giving products of coefficients less than 0.01 need to be computed.

Numbering the arguments N as in the first column of the tables, these two rules will be found to leave the following combinations as the only ones to be considered:

$$N_1 - N_2' = \theta \qquad N_2 - N_3' = \theta \qquad N_3 - N_1' = \theta \qquad N_7 + N_2' = \theta$$
$$N_8 + N_4' = \theta \qquad N_3 + N_1' = 2\pi - \theta \quad N_1 - N_4' = 2l' - \theta \quad N_8 + N_2' = 2l' - \theta$$

Using these numbers the computation of the formulæ (43) gives the following values of the coefficients k_a, k_e, etc., for the argument θ.

Arg.	k_l	k_π	k_θ	k_a	k_e	k_γ
$N_1 - N_2'$	+ .000358	0	+ 0.0891	+ 0.1779	− 0.0102	+ 1.972
$N_2 - N_3'$	+ 1	0	− 1	+ .0002	+ .0050	+ .003
$N_3 - N_1'$	0	0		0000	+ .0002	.000
$N_7 + N_2'$	+ 1	0	+ .0002	+ .0004	+ 0000	+ .013
$N_8 + N_4'$	0	0	− .0002	0000	0000	− .001
Sum	+ .000359	0	+ 0.0889	+ 0.1785	− 0.0050	+ 1.987

It will be remarked that the only accented coefficients are those of the last two lines; and that, as the combined argument $\theta - \Pi$ is the only one included in summation, the accented and unaccented k_a, etc., may all be combined.

For these numbers we derive by (45) and (46)

$$C_l = +0.3157 \qquad C_\pi = +0.0506 \qquad C_\theta = -11.067$$
$$C_a = +0.001393 \qquad C_e = +0.000187 \qquad C_\gamma = -0.4952$$

We have from the adopted elements of motion of the ecliptic:

$$\kappa = 0''.4714 \qquad \Pi = 173^\circ 30' + 0'.59(t - 1850)$$

The following are then the results for the argument θ:

$$\frac{\kappa}{\mathrm{N}} = -1''.396$$

$$\begin{array}{lll} \delta l_0 = -0''.441 \cos(\theta - \Pi) & \delta\pi_0 = -0''.01 \cos(\theta - \Pi) & \delta\theta_0 = +15''.45 \cos(\theta - \Pi) \\ \delta a = -0''.001945 \sin(\theta - \Pi) & \delta e = 0''.000 & \delta\gamma = +0''.691 \sin(\theta - \Pi) \end{array} \qquad (49)$$

To these expressions for δl_0, $\delta\pi_0$, and $\delta\theta_0$ are to be added the respective increments

$$\int \delta n dt \qquad \int \delta\pi_1 dt \qquad \text{and} \qquad \int \delta\theta_1 dt$$

arising from substituting the values of $\delta n (= -\frac{3}{2} n \delta a)$, $\delta e (= 0)$, and $\delta\gamma$ in the analytic expressions for n, π_1, and θ_1.

The value of δa gives the inequality of n

$$\delta n = -\tfrac{3}{2} n \delta a = 0''.00292 n \sin(\theta - \Pi)$$

This adds to the mean longitude the inequality

$$\delta l = -0''.00292 \nu \cos(\theta - \Pi)$$

where ν is the ratio $n : (\theta_1 - D_t \Pi) = -248.7$.

The complete inequality of the mean longitude thus takes the coefficient 0''.285.

We have from § 27, (74)

$$\delta\pi_1 = (.022 \delta a - .00433 \delta\gamma) n$$

The substitution of the preceding values of δa and $\delta\gamma$ gives the increments

$$\delta\pi_1 = -.00304 \sin(\theta - \Pi) \qquad \text{and} \qquad \delta\pi = -0''.76 \cos(\theta - \Pi)$$

We find, in the same way, the increment

$$\delta\theta = +0''.11 \cos(\theta - \Pi)$$

The inequalities of l, π, and θ now become

$$\delta l = +0''.285 \cos(\theta - \Pi) \quad \delta\pi = -0''.77 \cos(\theta - \Pi) \quad \delta\theta = +15''.56 \cos(\theta - \Pi) \qquad (50)$$

The coefficients of the arguments $2\pi - \theta$ and $2l' - \theta$ seem so small that we leave them out of consideration.

§ 65. *Inequalities arising from the coefficients E and F.*

These inequalities have been considered separately on account of their minuteness, and on their depending on arguments different from those of the other inequalities. Some special values of the coefficients E and F for Venus are given in tabular form in Table XII. In these expressions the axis of X passes through the mean Sun, as in the case of the inequalities depending on the mean longitudes. But, on essaying the computation of the principal inequalities arising from E and F, it was found that a fixed axis of X would be more convenient to use. The expressions were therefore transformed so as to refer them to the Sun's perigee as the initial axis. From the form of the expressions the equations of the transformation for x and y are readily found to be

$$x' = x\cos g' - y\sin g' \qquad y' = x\sin g' + y\cos g'$$

where the accents refer to the fixed solar perigee. It follows that if

$$E = a\cos N + b\sin N \qquad F = a'\cos N + b'\sin N$$

be any pair of the terms E and F depending on the argument N, the corresponding transformed terms, which we represent by E' and F', will be

$$\begin{aligned}E' = \tfrac{1}{2}(a+b')\cos(N+g') + \tfrac{1}{2}(a-b')\cos(N-g')&\\ +\tfrac{1}{2}(b-a')\sin(N+g') + \tfrac{1}{2}(b+a')\sin(N-g')&\\ F' = \tfrac{1}{2}(a'-b)\cos(N+g') + \tfrac{1}{2}(a'+b)\cos(N-g')&\\ +\tfrac{1}{2}(b'+a)\sin(N+g') + \tfrac{1}{2}(b'-a)\sin(N-g')&\end{aligned}$$

The transformed expressions thus arising are shown subsequently in Table XLII.

As a check against any large accidental error in the development of the coefficients, their approximate values, neglecting the small eccentricities of Venus and the Earth, were also computed by analytic development as follows: Taking the mean radius vector of the Earth as the unit of distance, and putting α for the corresponding numerical expression for the radius vector of Venus, the Laplace-Gauss form of development will give

$$\Delta^{-s} = \tfrac{1}{2}\Sigma b_s^{(i)}\cos iL$$

L being the difference of the heliocentric longitudes of Venus and of the Earth which we represent for the present by l and l' respectively.

The expressions for the rectangular geocentric coördinates of Venus will then be, when powers of the eccentricities and inclination are dropped in the development

$$\mathrm{X} = -\cos l' + \alpha\cos l \qquad \mathrm{Y} = -\sin l' + \alpha\sin l \qquad \mathrm{Z} = \alpha\sin I\sin(l - \theta_v)$$

where I is the inclination of the orbit of Venus, and θ_v the longitude of its node, reckoned from an arbitrary fixed origin. Forming the product of the several factors which form E and F, noting that the summation changes from positive to negative, changing and transforming the indices so as to reduce the summation to its simplest form, the values of E and F take the following general form:

$$E = \frac{\sin I}{4}\Sigma\{(ab_5^{(i+1)} - a^2b_5^{(i)}) \sin (iL + \theta_v) + (ab_5^{(i+1)} - a^2b_5^{(i+2)}) \sin (iL - 2l' + \theta_v)\}$$

$$F = \frac{\sin I}{4}\Sigma\{-(ab_5^{(i+1)} - a^2b_5^{(i)}) \cos (iL + \theta_v) + (ab_5^{(i+1)} - a^2b_5^{(i+2)}) \cos (iL - 2l' + \theta_v)\}$$

If we put, for brevity

$$\beta_i = \tfrac{1}{4}(ab_5^{(i+1)} - a^2b_5^{(i)}) \sin I \qquad \beta_i' = \tfrac{1}{4}(ab_5^{(i+1)} - a^2b_5^{(i+2)}) \sin I$$

we shall have

$$E = \Sigma\beta_i \sin (iL + \theta\ ) + \Sigma\beta_i' \sin (iL - 2l' + \theta_v)$$
$$F = -\Sigma\beta_i \cos (iL + \theta_v) + \Sigma\beta_i' \cos (iL - 2l' + \theta_v)$$

The numerical values of the coefficients $b^{(i)}$ may be taken from any one of various publications. In *Astronomical Papers of the American Ephemeris*, Vol. V, Pt. IV, p. 343, are found values of $c_5^{(i)} = a^2b^{(i)}$ for Venus and the Earth, as follows:

$i =$	0	1	2	3	4
$c_5^{(i)} =$	44.88	43.64	40.61	36.52	31.99

From these we find:

$i =$	-2	-1	0	$+1$	$+2$
$\frac{1}{4}ab_5^{(i+1)} =$	15.10	15.53	15.10	14.05	12.64
$\frac{1}{4}a^2b_5^{(i)} =$	10.15	10.91	11.22	10.91	10.15
$\beta_i \div \sin I =$	4.95	4.62	3.88	3.14	2.51
$\beta_i' \div \sin I =$	3.88	4.62	4.95	4.92	4.64

We thus have the following general expressions for E and F, the axis of X, in the ecliptic, being arbitrary.

We use

$$\sin I = .0592$$

Then

$$\begin{aligned}
E = &+ .293 \sin (-2L + \theta_v) + .229 \sin (-2L - 2l' + \theta_v) \\
&+ .273 \sin (-L + \theta_v) + .273 \sin (-L - 2l' + \theta_v) \\
&+ .229 \sin \theta_v \qquad\qquad\ + .293 \sin (-2l' + \theta_v) \\
&+ .186 \sin (L + \theta_v) \qquad + .291 \sin (-L - 2l' + \theta_v) \\
&+ .148 \sin (2L + \theta_v) \qquad + .274 \sin (2L - 2l' + \theta_v) \\
F = &- .293 \cos (-2L + \theta_v) + .229 \cos (-2L - 2l' + \theta_v) \\
&- .273 \cos (-L + \theta_v) + .273 \cos (-L - 2l' + \theta_v) \\
&- .229 \cos \theta_v \qquad\qquad\ + .293 \cos (-2l' + \theta_v) \\
&- .186 \cos (L + \theta_v) \qquad + .291 \cos (L - 2l' + \theta_v) \\
&- .148 \cos (2L + \theta_v) \qquad + .274 \cos (2L - 2l' + \theta_v)
\end{aligned}$$

Measuring θ_v from the solar perigee, in longitude 279°.5, we have

$$\theta_v = 155^\circ.4 \qquad l' = g' + 180^\circ$$

The results both of this computation and of the analytic development are shown in tabular form as follows:

TABLE XLII.

E AND *F* FOR THE ACTION OF VENUS.

Arg.	*E*				Arg.	*F*			
L, g'	cos		sin		L, g'	cos		sin	
	Num.	Anal.	Num.	Anal.		Num.	Anal.	Num.	Anal.
0 0	+.088	+.095	.000	.000	0 0	+.197	+.208	.000	.000
0 1	+.007	———	+.008	———	0 1	+.014	———	−.005	———
0 2	+.124	+.122	+.259	+.266	0 2	−.255	−.266	+.116	+.122
1 −2	+.120	+.121	−.252	−.264	1 −2	−.252	−.264	−.113	−.121
1 −1	+.012	———	−.006	———	1 −1	+.020	———	+.011	———
1 0	+.183	+.191	+.078	+.078	1 0	+.398	+.417	−.034	−.036
1 1	+.010	———	+.014	———	1 1	+.010	———	−.008	———
1 2	+.113	+.114	+.237	+.248	1 2	−.239	−.248	+.104	+.114
2 −2	+.112	+.114	−.236	−.249	2 −2	−.236	−.249	−.104	−.114
2 −1	+.010	———	−.002	———	2 −1	+.022	———	−.008	———
2 0	+.174	+.183	+.129	+.132	2 0	+.382	+.401	−.058	−.060
2 1	+.010	———	+.014	———	2 1	+.006	———	−.008	———
2 2	+.096	+.095	+.198	+.208	2 2	−.200	−.208	+.087	+.095

The largest terms arising from E and F are those whose arguments are independent of the mean longitude of the Moon, Sun, and Earth. These arise from the constant terms of E and F, which are, when referred to the solar perigee

$$E = +.088 \qquad F = +.197$$

The computation of the inequalities arising from this pair of terms will be yet further simplified by taking the node of Venus as the axis of X. By transforming to this axis we shall have

$$E = .002 \qquad F = -.216$$

We may regard this value of E as evanescent, thus confining the terms we have to determine to the expression

$$2F\eta\zeta$$

From the expressions for ξ, η, and ζ we find the largest terms of the products and their derivatives to be:

$$\begin{array}{rlll}
2\xi\zeta = & -.0895 \sin\theta & -.0039 \sin(2l' - \theta) & +.0006 \sin(2\pi - \theta) \\
2D'\xi\zeta = & -.1786 \quad `` & -.0135 \quad `` & +.0012 \quad `` \\
\dfrac{2\partial\xi\zeta}{\partial e} = & -.0137 \quad `` & +.0006 \quad `` & +.0218 \quad `` \\
\dfrac{2\partial\xi\eta}{\partial\gamma} = & -1.988 \quad `` & -.089 \quad `` & +.014 \quad ``
\end{array}$$

$$\begin{array}{rllll}
2\eta\zeta = & +.0895 \cos\theta & +.0039 \cos(2l'-\theta) & -.0006 \cos(2\pi-\theta) \\
2D'\eta\zeta = & +.1786 \text{ ``} & +.0135 \text{ ``} & -.0012 \text{ ``} \\
2\dfrac{\partial\eta\zeta}{\partial e} = & +.0137 \text{ ``} & -.0006 \text{ ``} & -.0218 \text{ ``} \\
2\dfrac{\partial\eta\zeta}{\partial\gamma} = & +1.988 \text{ ``} & +.089 \text{ ``} & -.014 \text{ ``}
\end{array}$$

The resulting terms of H, heretofore omitted, are

$$H = 2E\xi\zeta + 2F\eta\zeta$$

Taking the node of Venus as origin, we have, as shown on p. 135, the following terms of E and F

$$E = .285 \sin 2l' \qquad F = -.216 + .285 \cos 2l'$$

With these numbers we find for argument θ

$$H = -.0182 \cos\theta \qquad D'H = -.0348 \cos\theta \qquad \frac{\partial H}{\partial e} = -.0032 \cos\theta$$

$$\frac{\partial H}{\partial\gamma} = -.403 \cos\theta \qquad \frac{\partial H}{\partial l} = \frac{\partial H}{\partial\pi} = 0 \qquad \frac{\partial H}{\partial\theta} = +.0182 \sin\theta$$

These derivatives are to be substituted in the fundamental equations (41) and (42), § 21, and each equation integrated. For the latter process the factor of integration is

$$\frac{n}{\theta_1} = -248.8$$

The product of this into M for Venus (§ 17) is $-1''.055$

We thus have the following results:

$$\left.\begin{array}{lll}
D_{nt}l_0 = +.061M\cos\theta & D_{nt}\pi_0 = -.070M\cos\theta & D_{nt}\theta_0 = -2.24M\cos\theta \\
\delta l_0 = -0''.064 \sin\theta & \delta\pi_0 = +''.074 \sin\theta & \delta\theta_0 = +2''.36 \sin\theta \\
D_{nt}a = +.000136M\sin\theta & D_{nt}e = +.000047M\sin\theta & D_{nt}\gamma = -.1014M\sin\theta \\
\delta a = +0''.000144 \cos\theta & \delta e = +''.000050 \cos\theta & \delta\gamma = -0''.107 \cos\theta
\end{array}\right\} \quad (51)$$

To find the complete inequalities in l, π, and θ we must add the respective quantities

$$\int\delta n dt \qquad \int\delta\pi_1 dt, \qquad \int\delta\theta_1 dt$$

of which the expressions in terms of δn, δe, and $\delta\gamma$ are formed by § 27, Eq. 74. We thus have, dropping unimportant terms,

$$\delta n = -\tfrac{3}{2}n\delta a = -0''.000216n \cos\theta \qquad \delta\pi_1 = -0.0148\delta n - 0.0043n\delta\gamma = +0''.00046n \cos\theta$$

$$\delta\theta_1 = +.0038\delta n + .00066n\delta\gamma = -.000071n \cos\theta$$

The completed values of δl, $\delta\pi$, and $\delta\theta$ thus become

$$\begin{aligned} \delta l &= +0''.054 \sin\theta + \delta l_0 = -0''.010 \sin\theta \\ \delta\pi &= -0.115 \sin\theta + \delta\pi_0 \ = -0''.041 \sin\theta \\ \delta\theta &= +0.018 \sin\theta + \delta\theta_0 \ = +2''.38 \sin\theta \end{aligned} \tag{52}$$

in all which expressions θ is reckoned from the ascending node of Venus.

The coefficients of the term in $2\pi - \theta$ are, for δy and $\delta\theta$, less than one hundredth those for θ, and the integrating factor ν is less than 0.3 as great. The coefficients in $2l' - \theta$ are but a fraction of those in θ, and the integrating divisor is nearly 40 times as great. We therefore conclude that the inequalities depending on these arguments are inappreciable.

§ 66. *Action of Mars and Jupiter.* In Mars the product $M \sin I$ is about .08 that for Venus. I have therefore not computed the terms.

In the case of Jupiter the largest quantities which enter into the constant part of F are

$$\frac{a_1^5}{\Delta^5} = 1.26 \qquad YZ = \tfrac{1}{2}a_1^2 \sin I \qquad \sin I = 0.0231$$

Hence

$$a'^3F = 1.26 \times .0231\alpha^3 = +.000103$$

The product $10^8 MF$ is, approximately,

For Venus $-0''.92$ For Jupiter $+0''.170$

The inequalities depending on θ are proportional to this product. We conclude that the inequalities arising from the action of Jupiter may be derived from those of Venus by multiplying the coefficients by -0.185. We thus have, from the action of Jupiter,

$$\delta\theta = -0''.43 \sin(\theta - \theta_J) \qquad \delta\gamma = +0.020 \cos(\theta - \theta_J) \tag{53}$$

where θ_J is the longitude of the ascending node of Jupiter on the ecliptic. The inequalities of the other elements are unimportant.

§ 67. *Combination of terms depending on the longitude of the Moon's Node.* The inequalities (49), (50), (51), (52), and (53), all depending on the same argument θ, may now be combined. We shall do this for the two epochs, 1800 and 1900.

The value of Π which I have derived in *Elements and Constants*, p. 186, there called L', is

$$\Pi = 173^\circ 29'.7 + 54'.4T \text{ (from 1850)}$$

Taking approximate values of the nodes of Jupiter and Saturn, and this value of Π, we have

	1800	1900
Π	173° 2′	173° 57′

We take the nodes of Venus and Jupiter as constant, using the values for 1850

$$\theta_{\mathrm{v}} = 75^\circ.3 \qquad \theta_{\mathrm{J}} = 98^\circ.9$$

Carrying these values into the inequalities of the elements in question and combining them, we find:

$$\begin{aligned}
\delta l &= +\,0''.029 \sin\theta - 0''.271 \cos\theta \\
\delta\pi &= -\,0.10 \sin\theta + 0.80 \cos\theta \\
\delta\theta &= +\,2.55 \sin\theta - 17.33 \cos\theta \cdots \text{(for 1800)} \\
\delta\theta &= +\,2.31 \sin\theta - 17.34 \cos\theta \cdots \text{(for 1900)} \\
\delta\gamma &= -\,0''.114 \cos\theta - 0.769 \sin\theta \cdots \text{(for 1800)} \\
\delta\gamma &= -\,0.103 \cos\theta - 0.770 \sin\theta \cdots \text{(for 1900)}
\end{aligned}$$

§ 68. *Special computation of the Hansenian Venus-term of long period.*

The following are the planetary and lunar arguments whose differences make up the argument

$$18\mathrm{v} - 16g' - g$$

of the term in question.

	Planetary	Lunar
(1)	$18\mathrm{v} - 18g'$	$g - 2g'$
(2)	$18 \quad - 17$	$g - g'$
(3)	$18 \quad - 16$	g
(4)	$18 \quad - 15$	$g + g'$

The coefficients $h_{a,s}'$, $h_{a,c}'$, etc., are computed by §§ 22 and 23. The planetary coefficients *MK*, *MC*, and *MD* are found in Table X. The lunar coefficients *ap*, etc., are given in the next chapter, Tables XLVIII and XLIX. For the argument $g - g'$ we change the signs of a, e, g and *k*, as given for the argument $-g + g'$.

I have not computed the coefficients for the argument $g - 2g'$ believing their effect to be insensible. Their characteristic is $ee'^2 = .000050$, and, in the principal term of η, this is in Brown's theory multiplied by a factor of the order of magnitude .04. The largest planetary coefficient being $0.5 \div 10^3$, the value of h_a will be of the order of magnitude $1'' \div 10^9$, which would result in a term in δl of the order of magnitude 0''.02. Actually, the computation shows that the combinations (2) and (4) are also much smaller than (3).

We have now all the data for computing the coefficients $h_{a,c}'$, $h_{a,s}'$, etc., from the formulæ of § 22. The results are:

$$\begin{aligned}
h_{a,c}' &= -\,''.5597 \div 10^6 & h_{a,s}' &= +\,''.4880 \div 10^6 \\
h_{e,c}' &= -\,.0052 \div 10^3 & h_{e,s}' &= +\,.0047 \div 10^3 \\
h_{l,c}' &= -\,.00081 \div 10^3 & h_{l,s}' &= -\,.00094 \div 10^3 \\
h_{\pi,c} &= +\,.084 \div 10^3 & h_{\pi,s}' &= +\,.095 \div 10^3
\end{aligned}$$

The coefficients for γ and θ are much smaller, and are omitted. The coefficients we have given correspond to the argument

$$N - N_4 = g + 16g' - 18\mathrm{v} = A$$

of which the annual motion is

$$\mathrm{N} = -4747''.8$$

giving

$$\nu = -3649$$

We therefore have the following inequalities in l_0, π, and e

$$\begin{aligned} \delta l_0 &= -0''.003 \sin A + 0''.003 \cos A \\ \delta\pi &= +0\ .31\ \sin A - 0\ .35\ \cos A \\ \delta e &= -0\ .019 \sin A - 0\ .017 \cos A \end{aligned}$$

The term of δl_0 is so minute as to be unimportant. For the term in the mean longitude arising from δn we have

$$\log \tfrac{3}{2}\nu^2 = 7.3005$$

which gives

$$\delta l = -11''.18 \cos A + 9''.75 \sin A = 14''.83 \sin (A - 48^\circ 55'.2)$$

It will be convenient to use the negative of this argument in order that its motion may be positive. We shall therefore write

$$\delta l = 14''.83 \sin (18\mathrm{v} - 16g' - g + 228^\circ 55'.2)$$

where v is the mean longitude of Venus measured from the earth's perihelion.

It will be of interest to compare this result with those reached by other investigators. The following are arranged in the order of time. Putting

L, the mean long. of Venus—that of Earth

$$M = 18L + 2g' - g$$

and reducing all results to the mass 1÷408,000 of Venus, there has been found, for the direct action, by

Hansen *	$\delta l = 15''.34 \sin (M + 229^\circ.2)$
Delaunay †	$= 16\ .34 \sin (M + 228\ .5)$
Newcomb ‡	$= 14\ .80 \sin (M + 229\ .5)$
Radau §	$= 14\ .14 \sin (M + 229\ .0)$
Newcomb (above)	$= 14\ .83 \sin (M + 228\ .9)$

To judge the precision of this value we have to estimate the error to which the development by mechanical quadratures is liable. The circle being divided into 60 parts, any coefficient which we have taken as A_{18} is really the sum of an infinite series of which the first two terms are $A_{18} \pm A_{42}$. We have dropped all the terms after the first. From the progression of the coefficients it would seem that the

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* *Tables de la Lune*, p. 9.

† *Conn. des Temps*, 1862, App., p. 58.

‡ *Action of Planets*, p. 286.

§ *Inégalités Planétaires*, p. 113.

ratio $A_i : A_{i+1}$ is approximately 1 : 1.26, whence the ratio $A_{18} : A_{42}$ would be about 250. The error of the computed term may therefore well be ± 0″.06. It has been only as this work is in press that the author has looked into the possible effect of the slow convergence; and while it seems likely that the error entering through the coefficients K and C will not exceed that just stated, the same may not be true of the coefficient D.

A quantitative estimate of the correction may be made in various ways; but the author is unable to enter upon the subject in the present work.

It is also to be noted that the term as above computed contains the effect, whatever it may be, of the mutual perturbations of Venus and the Earth. A separate computation has been made of the fundamental numbers due to these perturbations, but as the final result of the coefficients amounts to only a fraction of a second, the computation has not been completed. The effect being included in the computed term, a knowledge of its amount is necessary to compare the result with that reached by the ordinary method of development.

The change in the term as computed is too minute to account for the observed variation of long period in the Moon's mean motion. As the period of this variation seems to be nearly the same as that of the inequality under consideration, the question naturally arises whether the effect of the indirect action may be appreciable. This being the most important question in the lunar theory, a computation of the principal part of the indirect term has been made. The result being altogether unimportant, it seems unnecessary to do more than present such a brief statement of the method as will enable the subject to be taken up by another in case the author's conclusion is not well founded. The required perturbations of the Earth by Venus are most easily computed for the case in question by using, instead of the Lagrangian brackets, the corresponding functions of the coördinates. The formulæ necessary for the purpose are found in Moulton's *Celestial Mechanics*, p. 291. The eccentricities have been dropped as unnecessary, and attention was confined to the longitude elements. The terms dependent upon the action of the planet on the Sun are also dropped, being appreciable only in terms depending on small multiples of mean longitude. The development of $\frac{a'^3}{\Delta^3}$ used in the computation is that in *Action*, pp. 248–251. The result for the indirect action is

$$\delta l = +\,0''.044 \cos A - 0''.036 \sin A.$$

This, being added to the terms already found, gives for the entire term

$$\delta l = 14''.77 \sin (18v - 16g' - g + 228^\circ\ 54')$$

which is the definitive result of the present investigation.

§ 69. *The Radau terms of long period.* Radau has computed certain additional terms of long period due to the action of Venus, with the following results, the arguments being reduced to those adopted in the present work:

$$\begin{aligned}
\delta v = &+0''.140 \sin (2\pi + g - 20\mathrm{v} + 19g' + 171^\circ) &\text{Per.} = 34^{\mathrm{y}}.8\\
&+0\ .110 \sin (g - 26\mathrm{v} + 29g' + 62^\circ) &127.2\\
&+0\ .056 \sin (g - 21\mathrm{v} + 21g) &8.35\\
&+0\ .019 \sin (\pi + g - 23\mathrm{v} + 24g' + 295^\circ) &55.\\
&+0\ .016 \sin (\pi + g - 15\mathrm{v} + 11g' + 219^\circ) &71.\\
&-0\ .012 \sin (2\pi - g + 24\mathrm{v} - 26g' + 159^\circ) &58.\\
&+0\ .012 \sin (g - 23\mathrm{v} + 24g' + 14^\circ) &7.6\\
&+0\ .008 \sin (\pi - \theta + g - 23\mathrm{v} + 24g' + 101^\circ) &28.2\\
&+0\ .004 \sin (2\theta - g + 23\mathrm{v} - 24g + 183^\circ) &42.\\
&+0\ .003 \sin (\pi - g + 21\mathrm{v} - 21g' + 288^\circ) &148.
\end{aligned}$$

The first three of these terms are the only ones that need be considered for the practical applications of the lunar theory. The third might also be omitted, but is easily computed in connection with the first.

For all the terms except the second the planetary coefficients A, B, C, and D may be derived with all necessary precision from the special values of these coefficients given in Table VII, by the following process. Putting

$$L = \mathrm{v} - g'$$

let the value of the planetary arguments for which we desire the coefficients be

$$N = hL + kg'$$

Recalling that the 720 special values of each coefficient, say A, are arranged in 12 systems of 60 indices each, the special value of N corresponding to the jth system and the index i will be

$$N_{ij} = 6^\circ \times hi + 30^\circ \times kj$$

We may mark each special value of A in the same way. The values of the coefficients A_c and A_s will then be given by the equations

$$360A_c = \Sigma A_{i,j} \cos N_{i,j}$$

$$360A_s = \Sigma A_{i,j} \sin N_{i,j}$$

The terms of A for the special argument N will then be

$$A = A_c \cos N + A_s \sin N$$

In most cases the computation may be simplified, as in the usual method of executing periodic developments, by adding together in advance the special values of A which are to be multiplied by the same sine or the same cosine. Another method may be used in computing these terms by the developments found in *Action*, Chapter III, § 18. Some modification is, however, necessary owing to the circumstance that in that work the rectangular coördinates are reckoned from a fixed axis passing through the earth's perihelion or the solar perigee, while in the present case the axes pass through the mean sun. It is therefore necessary to use the expressions for the geocentric coördinates of Venus referred to this moving axis, a development which may readily be made from the special values already given for the coördinates of Venus and the sun. It is necessary to transform the table so that the arguments shall be the mean anomaly of Venus instead of its mean longitude because the development for Δ^{-5} which are tabulated on p. 25 of *Action* have the mean anomaly of Venus as an argument.

I have applied the method of development from special values to the first term with the following results:

Planetary Coefficients for Arg. $20\mathrm{v} - 21g'$.

$A_0 = +.03562$ $\qquad A_1 = +.00694$
$B_0 = -.02999$ $\qquad B_1 = -.00583$
$C_0 = -.00563$ $\qquad C_1 = -.00112$
$D_0 = +.00634$ $\qquad D_1 = -.03254$
$K_0 = +.03280$ $\qquad K_1 = +.00638$
$C_0 = -.00282$ $\qquad C_1 = -.00056$

The lunar portion of the argument is equivalent $2D - g$, of which the indices in Table XL are $(-1, 0, 2, -2)$. From the numbers in this table we find for the direct action

$$\delta l = 0''.11 \sin(g + 2\pi - 20\mathrm{v} + 19g' + 10^\circ)$$

π being measured from the earth's perihelion.

This coefficient is less than that found by Radau; but the lunar argument is one to which the present method is not well adapted and a redetermination is desirable.

None of the other Radau terms are completely computed in the present work. Such computations as I have made seem to indicate even smaller coefficients than those found by Radau.

CHAPTER IX.

PERIODIC INEQUALITIES IN GENERAL.

§ 70. For convenience we mention the formulæ derived in Part I, giving them the special form adopted in the actual numerical work. We recall that the combination of any lunar argument N with a planetary argument N_4 gives rise to two arguments G, $N+N_4$ and $N-N_4$. For each argument there are two terms in the D_{nt} of each of the elements, one a cosine term; the other a sine term. We represent the coefficients of these terms for the element α by

$$h_{\alpha,c},\ h_{\alpha,s},\ h_{\alpha,c}',\ \text{and}\ h_{\alpha,s}'$$

with a similar notation for the remaining elements,

$$e,\ \gamma,\ l_0,\ \pi_0,\ \text{and}\ \theta_0$$

except that the coefficients for the angular elements have the negative sign.

The expressions of these coefficients for the direct action are given *in extenso* in Part I by the equations (46), (47), (48), (50), and (51). For the indirect action the coefficients are given in (64) and (65), but we may use the equations for direct action by making the substitution indicated in § 25 (66), which gives the expressions for the sum of the two actions.

For convenience in computation the coefficients are so used as to give the result in terms of 0″.001 as the unit. The numerical values of the planetary coefficients practically used for the purely periodic inequalities are these

$$K_c' = 10^3(MK_c - m^2G_c) \qquad C_c' = 10^3(\tfrac{1}{2}MC_c + m^2J_c) \qquad D_c' = 10^3(MD_c + m^2I_c)$$

with corresponding values of K_s', C_s', and D_s'.

Since each combination of a lunar with a planetary argument gives rise to two combined arguments, one equal to their sum the other equal to their difference, the coefficients relating to the latter are distinguished by accents.

The numerical values of the planetary coefficients, as derived from the numbers of Part II, and just defined, are shown in the following tables.

Values of the Planetary K′-Coefficients, Combining Direct and Indirect Action.

TABLE XLIII.

Action of Venus.

Arg. v, g'	K'_c	C'_c	D'_c	K'_s	C'_s	D'_s
	″	″	″	″	″	″
1 − 2	− 0.61	− 0.37	− 0.02	−0.11	.00	− 0.46
1 − 1	+25.04	−10.73	+35.66	+0.03	−.01	− 0.05
1 0	+ 2.44	− 0.52	+ 2.31	−0.02	+.03	− 0.04
2 − 4	− 0.68	− 0.02	+ 0.61	−0.06	−.06	− 0.12
2 − 3	+ 6.53	− 1.74	+20.01	+1.21	−.47	− 4.33
2 − 2	−29.33	+ 8.77	−52.76	+0.25	−.12	− 0.27
2 − 1	− 3.49	+ 0.50	− 3.29	−0.02	+.03	− 0.08
3 − 6	− 0.31	+ 0.02	+ 0.28	−0.24	+.02	− 0.22
3 − 5	− 0.94	+ 0.15	+ 6.54	−0.61	+.15	− 5.29
3 − 4	+10.08	− 3.25	+12.11	+2.25	−.69	− 2.69
3 − 3	+ 4.64	− 1.99	−12.92	+0.19	−.03	− 0.16
4 − 6	+ 0.61	− 0.20	+ 0.85	+0.53	−.19	− 0.76
4 − 5	+ 0.08	+ 0.03	− 2.19	+0.05	.00	+ 0.40
4 − 4	+ 7.11	− 2.38	− 9.72	−0.02	+.01	+ 0.08
5 − 8	+ 0.12	− 0.04	+ 0.71	+0.17	−.05	− 1.04
5 − 7	− 0.43	+ 0.16	− 0.88	−0.36	+.12	+ 0.78
5 − 6	+ 0.82	− 0.23	− 1.36	+0.09	−.06	+ 0.37
5 − 5	+ 8.40	− 2.43	− 7.39			
6 − 9	+ 0.02	− 0.01	− 0.02	+0.03	−.01	+ 0.03
6 − 8	+ 0.18	− 0.06	− 0.16	+0.14	−.05	+ 0.12
6 − 7	+ 1.19	− 0.28	− 0.11	+0.24	−.06	+ 0.21
6 − 6	+ 7.32	− 1.86	− 6.64	0.00	.00	0.00
7 − 9	+ 0.18	− 0.05	− 0.16	+0.12	−.04	+ 0.11
7 − 8	+ 1.13	− 0.24	− 0.11	+0.22	−.05	+ 0.21
7 − 7	+ 6.25	− 1.42	− 5.79	0.00	.00	0.00
8 −13	− 0.0045	+ 0.003	− 1.289	+0.0661	−.022	−15.92

Note.— The units in these tables are 0″.001.

TABLE XLIV.

ACTION OF MARS.

Arg. M, g'	K'_s	C'_c	D'_s	K'_s	C'_s	D'_c
1 −1	− 0″.78	+0″.13	+ 2″.34	−0″.01	+″.01	−0″.05
1 0	+ 0.18	−0.31	+ 0.32	+0.08	+.01	−0.27
2 −2	−11.70	+3.97	+17.04	+0.34	−.01	+0.26
2 −1	− 0.45	+0.38	−11.98	−0.63	+.24	+8.80
3 −3	+ 1.38	−0.43	− 0.95	+0.16	−.02	+0.15
3 −2	+ 0.81	−0.25	− 2.44	+0.89	−.27	+2.69
4 −4	+ 0.66	−0.17	− 0.20	+0.22	−.04	+0.21
4 −3	+ 1.86	−0.60	− 2.94	+1.84	−.64	+3.05
4 −2	− 0.12	+0.01	− 0.50	+0.59	−.25	−4.90
5 −4	− 0.52	+0.18	+ 0.34	−0.68	+.21	−0.48
5 −3	+ 0.04	−0.02	− 0.19	−0.47	+.15	−1.73
6 −5	− 0.22	+0.07	− 0.03	−0.33	+.09	−0.06
6 −4	+ 0.04	−0.02	− 0.11	−0.74	+.25	−1.34
6 −3	+ 0.15	−0.05	+ 0.69	−0.08	+.05	+0.50
15 −9	+ 0.11	−0.02	− 0.13	−0.15	+.04	−0.25
15 −8	+ 0.04	−0.02	− 1.55	0.00	+.01	−0.70

TABLE XLV.

ACTION OF JUPITER.

Arg. J, g'	K'_c	C'_c	D'_s	K'_s	C'_s	D'_c
+1 −2	− 4″.25	+ 0″.72	+ 4″.20	−0″.72	+0″.19	− 0″.43
+1 −1	−41.87	+12.87	+61.07	−1.53	+0.28	− 1.67
+1 0	+ 1.45	+ 0.21	+ 2.08	+1.35	−0.12	−21.72
+2 −3	+ 2.01	− 0.41	− 1.98	−0.20	+0.02	+ 0.10
+2 −2	+30.81	− 8.39	−17.37	+0.38	−0.06	+ 0.35
+2 −1	+ 7.86	− 2.88	−12.22	+3.26	−0.94	+ 5.29
+2 0	− 0.21	− 0.04	+ 0.53	−0.14	−0.01	+ 0.17
+3 −3	+ 4.53	− 0.63	−10.71	−0.40	+0.08	− 0.39
+3 −2	− 0.15	+ 0.06	+ 0.18	−5.93	+1.67	− 3.69
+3 −1	+ 0.24	− 0.08	− 0.40	−0.97	+0.38	− 1.67

TABLE XLVI.

ACTION OF SATURN.

Arg. S, g'	K'_s	C'_c	D'_s	K'_s	C'_s	D'_c
1 −1	−2″.46	+0″.83	+3″.48	−0″.03	+0″.01	−0″.03
1 0	+0.01	+0.03	−2.64	+0.01	−0.01	−0.50
2 −2	+1.33	−0.33	−1.57	0.00	0.00	0.00
2 −1	+0.66	−0.23	−0.94	+0.02	−0.01	+0.05

§ 71. The lunar coefficients fall into two classes, one determining the elements α, e, and γ and called, for brevity, the a-coefficients; the other determining l, π, and θ, and called the L-coefficients. Those of the first class are computed by the formulæ of § 20 and § 22; those of the second class by the formulæ of § 23, Eq. (50). In the computation we write k for k_4.

The a-coefficients are the nine products of the factors a, e, and g defined in § 22, Eq. 43, into p, q, and k.

From § 22, (46) to (48), it will be seen that by using the planetary factors in the form just given and taking the a-coefficients

$$\mathrm{a}p,\ \mathrm{a}q,\ \tfrac{1}{2}\mathrm{a}k, \text{ etc.}$$

the coefficients of the terms of $D_{nt}(\alpha$, e, and $\gamma)$ will each be the sum of three products of two factors each. But the quantities we actually compute are the values of $2\delta e$ and $2\delta\gamma$. We therefore double the coefficients for δe and $\delta\gamma$, using

$$2\mathrm{e}p,\ 2\mathrm{e}q,\ \mathrm{e}k;\ 2\mathrm{g}p,\ 2\mathrm{g}q \text{ and } \mathrm{g}k$$

We have also multiplied the inequalities of π and θ by the factors $2e$ and 2γ, required to reduce them to inequalities of the actual longitude and latitude. To do this we take for the nine L-coefficients

$$L',\ L'',\ \tfrac{1}{2}L_4;\ 2eP',\ 2eP'',\ eP_4;\ 2\gamma R',\ 2\gamma R'',\ \gamma R_4$$

Each of the coefficients to form a term of $D_{nt}l_0$, $2eD_{nt}\pi_0$ or $2\gamma D_{nt}\theta_0$ will then be the sum of three products formed by taking one factor from one of the Tables XLIII to XLVI, and the other from Table XLIX, the product $D'L_4$ being divided by 2.

TABLE XLVII.

Data for a-coefficients.

Arg.				Arg.				a	e	g
g	g'	λ	λ'		i	i'	i''			
0	0	0	0	0	0	0	0	0.0000	00.0000	00.0000
0	1	0	0	g'	0	0	0	0.0000	00.0000	00.0000
1	−1	0	0	$g-g'$	1	−1	0	+2.0529	+19.137	−00.0029
1	0	0	0	g	1	−1	0	+2.0529	+19.137	−00.0029
1	1	0	0	$g+g'$	1	−1	0	+2.0529	+19.137	−00.0029
2	0	0	0	$2g$	2	−2	0	+4.106	+38.274	−00.0058
−2	−0	2	−2	$2D-2g$	0	2	0	−0.0602	−38.307	−00.0400
−1	0	2	−2	$2D-g$	1	1	0	+1.9927	−19.170	−00.0429
0	0	2	−2	$2D$	2	0	0	+4.0456	−00.0336	−00.0458
1	0	2	−2	$2D+g$	3	−1	0	+6.098	+19.103	−00.0487
0	−1	2	−2	$2D-g'$	2	0	0	+4.046	−00.0336	−00.0458
0	1	2	−2	$2D+g'$	2	0	0	+4.046	−00.0336	−00.0458
−1	0	2	0	$2\lambda-g$	1	1	−2	+1.978	−19.175	+11.0971
0	0	2	0	2λ	2	0	−2	+4.0306	−00.0388	+11.0942
1	0	2	0	$2\lambda+g$	3	−1	−2	+6.083	+19.098	+11.0913
0	0	0	2	$2\lambda'$	0	0	−2	+0.015	− 0.005	+11.140

TABLE XLVIII.

LUNAR a-COEFFICIENTS FOR a, e, AND γ.

Arguments.								ap	aq	½ak	ep	eq	½ek	gp	gq	½gk
g, g', λ, λ'				l, π, θ, g'												
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	+1	−1	0	0	−0.032 09	−.111 47	−0.034 81	−0.299 1	−1.039 1	−0.324 6	+.000 05	+.000 16	+.000 05
2	0	0	0	+2	−2	0	0	−0.005 13	+.001 97	−0.004 99	−0.047 84	+0.018 37	−0.046 50	+.000 01	0	+.000 01
−1	1	0	0	−1	+1	0	+1	+0.000 13	−.000 73	+0.000 08	+0.001 24	−0.006 79	+0.000 77	0	0	0
1	1	0	0	+1	−1	0	+1	−0.001 39	+.000 56	−0.001 38	−0.012 92	+0.005 26	−0.012 82	0	0	0
−1	0	2	0	+1	+1	−2	0	−0.000 04	−.000 32	+0.000 10	+0.000 38	+0.003 17	−0.000 96	−.000 22	−.001 78	+.000 56
0	0	2	0	+2	0	−2	0	−0.000 77	+.024 10	−0.000 77	+0.000 01	−0.000 23	+0.000 01	−.002 11	+.066 34	−.002 11
1	0	2	0	+3	−1	−2	0	−0.000 06	+.001 99	−0.000 06	−0.000 19	+0.006 30	−0.000 19	−.000 11	+.003 66	−.000 11
−2	0	2	−2	0	+2	0	−2	−0.000 235	−.000 054	−0.000 216	−0.149 20	−0.034 28	−0.137 90	−.000 16	−.000 04	−.000 14
−1	0	2	−2	+1	+1	0	−2	−0.162 43	−.018 53	−0.162 95	+1.562 6	+0.178 3	+1.567 5	+.003 50	+.000 40	+.003 51
0	0	2	−2	+2	0	0	−2	+1.994 9	−.028 26	+1.994 70	−0.016 56	+0.000 23	−0.016 56	−.022 58	+.000 32	−.022 58
1	0	2	−2	+3	+1	0	−2	+0.166 0	−.001 28	+0.166 0	+0.519 98	−0.004 01	+0.519 98	−.001 32	+.000 01	−.001 32
0	1	2	−2	+2	0	0	−1	−0.006 43	+.000 06	−0.006 41	+0.000 05	$35 \div 10^8$	+0.000 05	+.000 07	$687 \div 10^8$	+.000 07
0	−1	2	−2	2	0	0	−3	+0.006 51	−.000 12	+0.006 35	−0.000 05	$708 \div 10^8$	−0.000 05	−.000 07	$137 \div 10^8$	−.000 07
0	0	0	2	0	0	−2	+2	$+297 \div 10^7$	$-63 \div 10^7$	$-297 \div 10^7$	$-99 \div 10^7$	$+21 \div 10^7$	$+99 \div 10^7$	+.022 06	−.004 68	−.022 06

TABLE XLIX.

LUNAR L-COEFFICIENTS FOR l, π, AND θ.

Arguments.								L'	L''	L_4	$2eP'$	$2eP''$	eP_4	$2\gamma R'$	$2\gamma R''$	γR_4
g, g' λ, λ				l, π, θ, g'												
0	0	0	0	0	0	0	0	−0.082 04	+2.004 31	……	−0.116 35	−0.181 96	……	−.003 85	+.135 12	……
1	0	0	0	+1	−1	0	0	−0.123 05	−0.199 14	−0.275 07	+0.595 03	+2.056 09	+0.640 4	−.002 46	−.012 85	−.002 50
2	0	0	0	+2	−2	0	0	−0.010 05	+0.002 48	−0.019 74	+0.077 07	+0.057 30	+0.072 8	−.000 01	……	−.000 04
−1	1	0	0	−1	+1	0	4	−0.000 32	−0.003 16	+0.002 44	0	+0.006 73	……	……	……	……
0	1	0	0	0	0	0	+1	−0.005 454	+0.001 241	−0.014 128	−0.004 743	−0.001 172	−0.006 53	−.000 073	+.000 041	−.000 12
1	1	0	0	+1	−1	0	+1	−0.005 59	+0.002 42	−0.010 86	+0.026 51	−0.000 39	+0.027 0	……	……	……
−1	0	2	0	+1	+1	−2	0	−0.000 09	+0.000 11	+0.000 38	……	+0.024 25	……	……	+.014 50	……
0	0	2	0	+2	0	−2	0	−0.000 99	+0.017 75	−0.002 03	+0.000 02	+0.003 15	+0.000 02	+.004 33	−.137 90	+.003 78
1	0	2	0	+3	−1	−2	0	−0.000 16	+0.001 05	−0.000 32	……	−0.008 24	……	……	−.004 90	……
−2	0	2	−2	0	+2	+0	−2	+0.013 67	+0.006 84	+0.023 51	−0.298 75	−0.069 08	−0.275 19	+.000 04	+.000 01	+.000 04
−1	0	2	−2	+1	+1	0	−2	−0.302 23	−0.070 22	−0.612 74	+3.117 20	+0.360 88	+3.117 4	−.005 00	−.001 36	−.005 03
0	0	2	−2	+2	0	0	−2	+1.992 22	−0.076 97	+3.984 14	+0.298 00	+0.022 69	+0.298 1	+.045 17	−.010 28	+.045 14
1	0	2	−2	+3	−1	0	−2	+0.102 62	−0.002 34	+0.205 24	−1.027 38	+0.007 90	−1.027 3	+.001 80	……	+.001 75
2	0	2	−2	+4	−1	0	−2	+0.005 12	−0.000 11	+0.010 24	−0.114 62	……	−0.114 6	+.000 01	……	+.000 01
−1	1	2	−2	+1	+1	0	−1	−0.003 614	−0.000 172	−0.007 273	+0.015 989	……	+0.015 785	−.000 003	……	−.000 002
0	1	2	−2	+2	0	0	−1	−0.009 22	+0.000 94	−0.018 40	+0.000 02	……	……	……	……	……
1	1	2	−2	+3	−1	0	−1	−0.001 58	+0.000 01	−0.003 16	+0.006 73	……	+0.006 3	……	……	……
−1	−1	2	−2	+1	+1	0	−3	+0.001 810	−0.003 176	+0.003 661	−0.009 992	+0.015 462	−0.010 1	+.000 002	−.000 003	+.000 001
0	−1	2	−2	2	0	0	−3	+0.009 43	−0.004 14	+0.018 66	−0.000 02	+0.000 01	−0.000 02	……	……	……
1	−1	2	−2	3	−1	0	−3	+0.002 27	−0.000 15	+0.004 54	−0.008 927	−0.004 64	−0.008 8	……	……	……
−1	0	0	2	−1	+1	−2	+2	−0.000 11	−0.000 07	−0.000 22	……	……	……	……	……	……
0	0	0	2	0	0	−2	+2	+0.005 65	−0.001 69	−0.011 29	−0.000 23	+0.000 04	+0.000 22	−.048 75	+.008 85	+.048 80
1	0	0	−2	+1	−1	−2	+2	−0.000 37	−0.000 95	+0.000 90	……	……	……	……	……	……
−2	−1	2	−2	……	……	……	……	−0.000 019	+0.000 130	−0.000 207	+0.000 915	−0.002 766	+0.002 976	0	0	0
−2	1	2	−2	……	……	……	……	+0.000 101	−0.000 006	+0.000 224	−0.002 797	−0.000 042	−0.002 966	0	0	0

§ 72. From these two tables the four coefficients for each element are formed by the following computation, an adaptation of (46) to (51)

The inequalities of e have received the factor 2, and those of π the factor $2e$ in order to transform them into the principal terms of the true longitude without further multiplication.

Two other points which may be recalled are these: (1) We use k instead of κ_4 in the formulæ; (2) it is to be recalled that C_c' and C_s' contain only $\frac{1}{2}C$, as that symbol is used in Part I.

Element a

$$a_1 = K_s'\mathrm{a}p - C_s'\mathrm{a}q \qquad a_2 = -K_c'\mathrm{a}p + C_c'\mathrm{a}q$$
$$h_{a,c} = \tfrac{1}{2}D_c'\mathrm{a}k + a_1 \qquad h_{a,c}' = \tfrac{1}{2}D_c'\mathrm{a}k - a_1$$
$$h_{a,s} = a_2 + \tfrac{1}{2}D_s'\mathrm{a}k \qquad h_{a,s}' = a_2 - \tfrac{1}{2}D_s'\mathrm{a}k$$

Element e

$$c_1 = 2K_s'\mathrm{e}p - 2C_s'\mathrm{e}q \qquad e_2 = -2K_c'\mathrm{e}p + 2C_c'\mathrm{e}q$$
$$2h_{e,c} = D_c'\mathrm{e}k + e_1 \qquad 2h_{e,c}' = D_c'\mathrm{e}k - e_1$$
$$2h_{e,s} = e_2 + D_s'\mathrm{e}k \qquad 2h_{e,s}' = e_2 - D_s'\mathrm{e}k$$

Element γ

$$\gamma_1 = 2K_s'\mathrm{g}p - 2C_s'\mathrm{g}q \qquad \gamma_2 = -2K_c'\mathrm{g}p + 2C_c'\mathrm{g}q$$
$$2h_{\gamma,c} = D_c'\mathrm{g}k + \gamma_1 \qquad 2h_{\gamma,c}' = D_c'\mathrm{g}k - \gamma_1$$
$$2h_{\gamma,s} = \gamma_2 + D_s'\mathrm{g}k \qquad 2h_{\gamma,s}' = \gamma_2 - D_s'\mathrm{g}k$$

Element l_0

$$\lambda_1 = K_c'L' - C_c'L'' \qquad \lambda_2 = K_s'L' - C_s'L''$$
$$h_{l,c} = \lambda_1 - \tfrac{1}{2}D_s'L_4 \qquad h_{l,c}' = \lambda_1 + \tfrac{1}{2}D_s'L_4$$
$$h_{l,s} = \tfrac{1}{2}D_c'L_4 + \lambda_2 \qquad h_{l,s}' = \tfrac{1}{2}D_c'L_4 - \lambda_2$$

Element π_0

$$\pi_1 = 2K_c'eP' - 2C_c'eP'' \qquad \pi_2 = 2K_s'eP' - 2C_s'eP''$$
$$2eh_{\pi,c} = \pi_1 - D_s'eP_4 \qquad 2eh_{\pi,c}' = \pi_1 + D_s'eP_4$$
$$2eh_{\pi,s} = D_c'eP_4 + \pi_2 \qquad 2eh_{\pi,s}' = D_c'eP_4 - \pi_2$$

Element θ_0

$$\theta_1 = 2K_c'\gamma R' - 2C_c'\gamma R'' \qquad \theta_2 = 2K_s'\gamma R' - 2C_s'\gamma R''$$
$$2\gamma h_{\theta,c} = \theta_1 - D_s'\gamma R_4 \qquad 2\gamma h_{\theta,c}' = \theta_1 + D_s'\gamma R_4$$
$$2\gamma h_{\theta,s} = D_c'\gamma R_4 + \theta_2 \qquad 2\gamma h_{\theta,s}' = D_c'\gamma R_4 - \theta_2$$

In the exceptional cases when one of the constituent factors of either class, planetary or lunar, is a constant, there will be a merging of the accented and unaccented arguments and terms.

For the case $N = 0$, a, e and g all vanish, and we have

$$D_{nt}a = D_{nt}e = D_{nt}\gamma = 0 \qquad k = 0$$

while (49) of § 23 may be written

$$-D_{nt}l_0 = (2K_c'L_0' - 2C_c'L_0'') \cos N_4 + (2K_s'L_0' - 2C_s'L_0'') \sin N_4$$

We have, therefore, in this case, only to double the values of the L-coefficients for argument 0.

In the combination of a constant planetary factor ($N_4 = 0$) with a periodic lunar factor we may use, instead of (46)

$$h_{a,s} = -2K_0'ap + 2C_0'aq$$

Then

$$\delta a = -\nu h_{a,s} \cos N$$

with similar equations for e and γ, formed by writing e and g for a. We also have, instead of (51)

$$h_{l,c} = 2K_0'L' - 2C_0'L''$$

Then

$$\delta l_0 = -\nu h_{lc,} \sin N$$

with similar equations for π and θ.

As neither D nor I has a constant term, there are only cosine-terms of this class in a, e, and γ, and only sine-terms in l, π, and θ.

From these coefficients for the D_{nt} of the elements we have those for the elements themselves by multiplication by the integrating factor ν. The motion of the lunar argument is

$$in + i'\pi_1 + i''\theta_1 + jn' = \mathrm{N}$$

and that of the planetary argument

$$k'n' + kn_4 = \mathrm{N}_4$$

We compute

$$\nu = \frac{n}{\mathrm{N} + \mathrm{N}_4} = \frac{1}{\mathrm{n} + \mathrm{n}_4}$$

$$\nu' = \frac{n}{\mathrm{N} - \mathrm{N}_4} = \frac{1}{\mathrm{n} - \mathrm{n}_4}$$

Then the coefficients which we compute are

$$a_s = \nu h_{a,c} \qquad a_c = -\nu h_{a,s} \qquad a_s' = \nu' h_{a,c}' \qquad a_c' = -\nu' h_{a,s}'$$

$$2e_s = 2\nu h_{e,c} \qquad 2e_c = -2\nu h_{e,s} \qquad 2e_s' = 2\nu' h_{e,c}' \qquad 2e_c' = -2\nu' h_{e,s}'$$

$$2\gamma_s = 2\nu h_{\gamma,c} \qquad 2\gamma_c = -2\nu h_{\gamma,s} \qquad 2\gamma_s' = 2\nu' h_{\gamma,c}' \qquad 2\gamma_c' = -2\nu' h_{\gamma,s}'$$

$$l_s = \tfrac{3}{2}\nu^2 h_{a,s} - \nu h_{l,c} = \nu(\tfrac{3}{2}\nu h_{a,s} - h_{l,c})$$

$$l_c = \tfrac{3}{2}\nu^2 h_{a,c} + \nu h_{l,s} = \nu(\tfrac{3}{2}\nu h_{a,c} + h_{l,s})$$

$$2e\pi_s = -\nu \times 2eh_{\pi,c}$$

$$2e\pi_c = +\nu \times 2eh_{\pi,s}$$

with similar forms for θ when required.

The inequalities of the elements are then

$$\begin{array}{lllll}
\delta l = l_c \cos (N + N_4) & + \, l_s \sin (N + N_4) & + \, l_c' \cos (N - N_4) & + \, l_s' \sin (N - N_4) & \\
\delta \pi = \pi_c \quad \text{``} & + \, \pi_s \quad \text{``} & + \, \pi_c' \quad \text{``} & + \, \pi_s' \quad \text{``} & \\
\delta \theta = e_c \quad \text{``} & + \, e_s \quad \text{``} & + \, e_c' \quad \text{``} & + \, e_s' \quad \text{``} &
\end{array}$$

A similar computation was made for γ and θ; but the results were unimportant in all but one of the arguments.

§ 73. The motions of the arguments from which the integrating factors ν or ν' are to be computed are the following. The sidereal motion for a Julian year is given in revolutions for the lunar, and in seconds for the planetary arguments. Then follows the ratio of each to the mean motion of the Moon.

Motions of Arguments.

	Mot. in $365^d.25$	n
$g;\ g_1 =$	$13^r.255\,523$	0.991 5452
$l;\ n =$	13 .368 513	1.
$\pi;\ \pi_1 =$	0 .112 990	0.008 4518
$\theta;\ \theta_1 = -$	0 .053 765	− 0.004 0218
Venus	2106 641″.38	0.121 5913
Earth	1295 977 .43	0.074 8013
Mars	689 050 .9	0.039 7707
Jupiter	109 256 .6	0.006 3061
Saturn	43 996 .2	0.002 5394

The elemental inequalities computed from these formulæ are shown in tabular form on the following pages. On making the computation it was found that the coefficients for a were so minute that no terms in the parallax would need to be considered, and only in some exceptional cases, generally terms of long period, did the inequality of γ affect the longitude. The coefficients for these elements are therefore omitted in the tables of longitude elements. The given coefficients are those for the mean longitude, δl, $2\delta e$, $e\delta\pi$. It must be remembered that the accented e' and π' do not refer to solar elements, but designate only the coefficients depending upon the differences between the lunar and the planetary arguments, while the unaccented coefficients depend upon their sum.

It was also found that the inequalities of γ and θ were insensible in nearly all cases. The few terms of these elements found to be sensible are therefore given separately.

§ 74. *Terms with purely Lunar Arguments.* We here make a single computation for the combined action of all the planets. To include the effect of the indirect action, we have only to modify the values of *MK*, etc., as indicated in (66).

Then, from the values of the constant term already given for the four principal disturbing planets in § 54 we find

$$
\begin{aligned}
10^3\Sigma MK_0 &= +6''.070 & \qquad 10^3\Sigma MC_0 &= -5''.76 \\
-10^3 m^2 G_0 &= -0.459 & 10^3 m^2 J_0 &= +0.153 \\
10^3 K_0' &= +5.611 & 10^3 C_0' &= -2.727
\end{aligned}
$$

For the terms in question we now have, for each lunar argument

$$
\begin{aligned}
h_{a,s} &= -11''.22\mathrm{a}p - 5''.45\mathrm{a}q \\
h_{e,c} &= -11''.22\mathrm{e}p - 5''.45\mathrm{e}q
\end{aligned}
$$

and

$$
\begin{aligned}
h_{l,c} &= +11''.22L' + 5''.45L'' \\
h_{\pi,c} &= +11''.22P' + 5''.45P''
\end{aligned}
$$

the terms in γ and θ being omitted as unimportant.

The inequalities of l_0, e, and π may now be computed as in §§ 26 and 27. The most condensed formulæ of computation are

$$
\begin{aligned}
10^3 l_s &= -\nu^2(16''.8\mathrm{a}p + 8''.2\mathrm{a}q) - \nu(11''.2L' + 5.4L'') \\
10^3 e\pi_s &= -\nu(11''.2eP' + 5.4eP'') \\
10^3 e_c &= +\nu(11''.2\mathrm{e}p + 5.4\mathrm{e}q)
\end{aligned}
$$

The elemental inequalities then are

$$\delta l = l_s \sin N \qquad e\delta\pi = e\pi_s \sin N \qquad \delta e = e_c \cos N$$

The results of this computation for the only terms which I have found to give any appreciable result are, in units of 0''.001;

Arg.	$10^3 l_s$	$10^3 e_c$	$10^3 e\pi_s$
g	3.8	− 9.0	− 9.0
$2D-2g$	+ 1.5	+ 14.0	− 13.6
$2D-2g$	+ 8.2	+ 21.6	− 21.5
$2D$	− 22.0	− 0.1	− 0.9
$2\lambda'$	+ 0.4	+ 2.0	+ 2.0

The only corrections of the true longitude to be considered are the following to the evection and variation.

$$\delta v = +0''.036 \sin(2D-g) + 0''.021 \sin 2D.$$

§ 75. *Elemental Inequalities.* The miscellaneous inequalities of the mean longitude, the eccentricity and the perigee, as given by the preceding formulæ and data, are tabulated in the following pages.

It may be repeated that the mean longitudes, V, M, J, and S, are measured from the solar perigee.

Periodic Elemental Inequalities in Units of 0.″001.

TERMS INDEPENDENT OF THE LUNAR ARGUMENTS. ($N = 0$.)

Action of Venus.

Arg.	ν	l_s	l_c	$2e\pi_s$	$2e\pi_c$
V	+8.22	−14	−1	+6	0
$V-g'$	+21.37	−831	+1	+208	0
$V-2g'$	−35.70	+56	−1	0	−1
$2V-2g'$	+10.69	+324	+5	−107	−1
$2V-3g'$	+53.25	−314	+90	+114	−24
$2V-4g'$	−17.85	+3	−4	+3	0
$3V-3g'$	+7.12	−51	+1	+13	0
$3V-4g'$	+15.25	−173	+36	+54	−12
$3V-5g'$	−108.3	−48	+54	+30	−21
$5V-8g'$	+105	−15	+18	+4	−6
$8V-13g'$	+319.4	+16	+246	−5	−74

Action of Mars.

Arg.	ν	l_s	l_c	$2e\pi_s$	$2e\pi_c$
M	+25.1	−30	−1	+4	−0
$M-g'$	−28.6	−11	+1	+7	0
$2M-g'$	+211.0	+308	−181	−48	+49
$2M-2g'$	−14.27	−200	0	+60	0
$3M-2g'$	−33.0	+29	−31	−9	+10
$3M-3g'$	−9.5	+14	0	−4	0
$4M-2g'$	+105.5	+2	+95	−3	−24
$4M-3g'$	−15.3	+32	−35	−10	+10
$5M-3g'$	−39.1	+23	−20	−1	−6
$15M-8g'$	−540.5	+39	+22	−9	−2

Action of Saturn.

Arg.	ν	l_s	l_c	$2e\pi_s$	$2e\pi_c$
S	+39.4	+48	+16	−4	0
$S-g'$	−13.8	−40	0	+11	0
$2S-g'$	−14.3	+11	−1	−3	0
$2S-2g'$	−6.9	+8	0	−3	0

Action of Jupiter.

Arg.	ν	l_s	l_c	$2e\pi_s$	$2e\pi_c$
J	+1586	+171	+41	+41	−57
$J-g'$	−14.6	−652	+13	+210	−7
$J-2g'$	−7.0	−15	+4	+9	−2
$2J-g'$	−16.1	+165	−52	−46	−18
$2J-2g'$	−7.30	+208	−1	−74	−1
$2J-3g'$	−4.7	+6	0	−3	0
$3J-g'$	−17.9	+5	−24	−2	−6
$3J-2g'$	−7.6	−2	+42	+1	−15
$3J-3g'$	−4.9	+9	+1	−6	+1

LUNAR ARGUMENT $N = g$.

Planetary Argument.	ν	ν'	l_s	l_c	l_s'	l_c'	$2e_s$	$2e_c$	$2e_s'$	$2e_c'$	$2e\pi_s$	$2e\pi_c$	$2e\pi_s'$	$2e\pi_c'$
$V-g'$	+0.9631	+1.0585	+1	−0	+16	0	0	−13	0	−63	−13	0	−63	0
$2V-3g'$	+0.9898	+1.0281	−4	+1	+6	+1	+1	+5	+5	−22	+5	−1	−22	−5
$2V-2g'$	+0.9216	+1.1137	−1	0	−22	0	0	+1	0	+77	+1	0	+77	0
$3V-5g'$	+1.0180	+0.9993	+1	+1	+1	+1	+3	+5	+3	−3	+5	−3	−3	−3
$3V-4g'$	+0.9460	+1.0800	0	0	+4	0	0	−5	+4	−23	−5	0	−23	−4
$3V-3g'$	+0.8814	+1.1750	+3	0	−1	0	0	−13	0	+1	−13	0	+1	0
$2M-2g'$	+1.0853	+0.9420	−6	0	+1	0	0	+29	0	+5	+29	0	+5	0
$2M-g'$	+1.0038	+1.0134	+2	0	−2	−7	−6	+9	−1	0	+7	+5	+9	+5
$J-g'$	+1.0834	+0.9434	−23	0	0	0	+2	+98	−1	+10	+97	−2	+11	0
J	+1.0022	+1.0150	0	+5	0	+13	+5	0	+15	−2	0	−13	−2	−15
$2J-2g'$	+1.1703	+0.8861	+150	0	+40	0	0	−54	0	−21	−54	0	−21	0
$2J-g'$	+1.0761	+0.9490	+40	−1	00	−8	−8	−21	+1	−3	−21	+8	−3	−1

LUNAR ARGUMENT $N = 2D - 2g = 2\pi - 2g'$ (EVECTION-TERMS).

Planetary Argument.	ν	ν'	l_s	l_c	l_s'	l_c'	$2e_s$	$2e_c$	$2e_s'$	$2e_c'$	$2e\pi_s$	$2e\pi_c$	$2e\pi_s'$	$2e\pi_c'$
V−g′	− 11.64	− 5.572	− 1	0	+ 6	0	0	− 19	0	+ 101	+ 18	0	− 101	0
2V−3g′	− 8.779	− 6.602	− 2	0	+ 3	+ 1	− 7	− 30	− 11	+ 50	+ 30	− 8	− 51	− 11
2V−2g′	− 25.57	− 4.419	+ 9	0	− 6	0	0	+134	− 1	− 106	−130	0	+ 107	− 1
3V−5g′	− 7.046	− 8.100	− 1	+ 1	+ 1	+ 1	−12	− 15	− 10	+ 12	+ 15	−12	− 13	− 11
3V−4g′	− 14.90	− 5.045	0	0	+ 2	0	0	− 1	− 7	+ 33	0	− 1	− 33	− 7
3V−3g′	+129.9	− 3.66	+72	0	0	0	− 2	−661	0	− 7	+659	− 2	+ 7	0
6V−8g′	−625.0	− 3.79	+50	−37	0	0	+49	+ 64	0	0	− 64	+49	0	0
5V−8g′	+737.7	− 3.43	0	0	0	0	− 5	− 24	0	0	+ 24	− 5	0	0
M−g′	− 5.962	− 10.24	0	0	0	0	0	− 5	0	+ 4	+ 5	0	− 4	0
2M−2g′	− 4.932	− 15.96	− 2	0	+ 1	0	+ 1	− 42	0	+ 15	+ 42	0	− 15	0
2M−g′	− 7.815	− 7.276	+ 1	− 2	− 5	− 2	+17	+ 25	+ 19	− 25	− 25	+17	+ 25	+ 19
3M−3g′	− 4.205	− 36.23	0	0	0	0	0	+ 3	0	+ 6	− 3	0	− 6	0
3M−2g′	− 6.135	− 9.766	0	0	0	0	+ 6	+ 6	+ 4	− 4	− 6	+ 6	+ 4	+ 5
4M−3g′	− 5.050	− 14.84	0	0	0	0	+ 7	+ 7	+ 4	− 3	− 7	+ 7	+ 4	+ 5
4M−2g′	− 8.116	− 7.034	0	0	0	0	+ 9	+ 1	− 11	− 1	− 1	−10	+ 1	− 11
15M−8g′	− 7.429	− 7.645	0	0	0	0	− 1	+ 3	− 1	− 3	− 1	−29	0	0
6M−5g′	− 3.73	+370.4	0	0	− 1	− 8	− 1	0	− 22	+ 29	+ 0	− 1	− 29	− 23
J−2g′	− 3.6232	+ 94.3396	0	0	0	− 1	− 1	− 9	− 10	+ 15	+ 9	− 1	− 16	− 9
J−g′	− 4.9702	− 15.5763	− 8	0	+ 3	0	− 5	−151	0	+ 54	+151	− 5	− 53	0
J	− 7.9114	− 7.1942	0	+ 2	0	+ 3	−44	− 1	− 46	+ 7	+ 1	−44	− 7	− 46
2J−3g′	− 2.9028	+ 12.6422	0	0	0	0	0	+ 3	− 1	− 1	− 3	0	0	− 1
2J−2g′	− 3.7078	+232.720	+ 3	0	+256	+ 1	+ 1	+ 54	+ 5	−1158	− 54	+ 1	+1164	+ 3
2J−g′	− 5.1308	− 14.1844	+ 2	− 1	− 1	0	+13	+ 30	+ 6	− 12	− 31	+13	+ 15	+ 6
3J−3g′	− 2.9568	+ 13.7363	+ 1	0	+ 1	0	+11	+ 13	0	+ 22	− 13	− 1	− 21	0
3J−2g′	− 3.7965	−497.760	0	+ 1	− 8	−258	−11	0	+429	− 14	0	−11	+ 15	+431
3J−g′	− 5.3022	− 13.0208	0	0	0	0	− 4	+ 1	− 2	0	− 1	− 4	+ 1	− 2

LUNAR ARGUMENT $N = 2D - g = g + 2\pi - 2g'$.

Planetary Argument.	ν	ν'	l_s	l_c	l_s'	l_c'	$2e_s$	$2e_c$	$2e_s'$	$2e_c'$	$2e\pi_s$	$2e\pi_c$	$2e\pi_s'$	$2e\pi_c'$
V−g′	1.1042	1.231	− 6	0	+47	0	0	− 33	0	+239	+ 32	0	−237	0
2V−3g′	1.1395	1.190	−10	+ 4	+20	+ 1	−10	− 48	−20	+100	+ 47	−11	− 99	−21
2V−2g′	1.0500	1.307	+10	0	−69	0	0	+ 75	− 3	−342	− 74	0	+341	− 3
3V−5g′	1.1770	1.152	− 4	+ 5	+ 4	+ 4	−21	− 28	−16	+ 21	+ 27	−22	− 20	−17
3V−4g′	1.0828	1.261	− 1	0	+18	+ 5	− 1	− 5	−19	+ 89	+ 5	− 1	− 89	−19
3V−3g′	1.008	1.392	+10	0	− 7	0	+ 1	+ 57	− 1	− 35	− 55	+ 1	+ 35	− 1
2M−2g′	1.2679	1.0765	−23	0	+ 3	0	+ 3	−116	0	+ 17	+121	0	− 16	0
2M−g′	1.1581	1.1708	+ 9	− 6	− 9	− 7	+30	+ 42	+35	− 46	− 42	+29	+ 44	+34
4M−3g′	1.2602	1.0821	+ 2	− 2	0	0	+20	+ 19	+ 4	− 3	− 19	+19	+ 3	+ 4
J−2g′	1.3976	0.9979	+ 9	− 6	− 9	0	− 5	− 38	+ 1	− 1	+ 36	− 5	0	+ 1
J−g′	1.2653	1.0784	−81	0	+10	0	−13	−416	0	+ 59	+415	−13	− 59	0
J	1.1559	1.1730	− 1	+16	+ 1	+16	−74	− 2	−84	+ 14	+ 1	−74	− 13	−84
2J−3g′	1.5456	0.9340	+ 3	0	0	0	− 2	+ 19	+ 1	0	− 19	− 2	0	+ 1
2J−2g′	1.3854	1.0042	+44	0	+ 8	0	+ 3	+215	0	+ 45	−211	+ 3	− 45	0
2J−g′	1.2553	1.0858	+15	− 5	− 4	− 4	+34	+ 81	+ 8	− 13	− 81	+34	+ 13	+ 8
3J−3g′	1.5307	0.9395	+17	0	− 3	0	− 3	+ 73	0	− 19	− 72	− 3	+ 18	0
3J−2g′	1.3734	1.0106	0	+10	0	− 1	−42	− 3	+ 7	0	+ 3	−42	0	+ 7
3J−g′	1.2455	1.0933	0	0	0	0	−10	+ 2	− 2	0	− 2	−10	0	− 2

LUNAR ARGUMENT $2D = 2g + 2\pi - 2g'$.

Planetary Argument.	ν	ν'	l_s	l_c	l'_s	l'_c	$2e_s$	$2e_c$	$2e'_s$	$2e'_c$	$2e\pi_s$	$2e\pi_c$	$2e\pi'_s$	$2e\pi'_c$
V−g′	+.527	+.554	+ 21	0	−122	0	0	0	0	−1	+ 2	0	−10	0
2V−2g′	+.514	+.569	− 43	0	+172	0	0	−1	0	+2	− 4	0	+14	0
2M−2g′	+.562	+.521	+ 59	+ 2	− 11	0	0	+1	0	−1	+ 5	0	− 1	0
J−g′	+.561	+.521	+212	− 6	− 36	0	0	+2	0	−1	+17	0	− 3	0
J	+.539	+.542	+ 1	−39	− 7	−45	+1	0	+1	0	+ 1	−4	− 1	−4
2J−2g′	+.584	+.503	−106	+ 2	− 23	0	0	−2	0	0	− 9	0	− 3	0

LUNAR ARGUMENT $2D + g = 3g + 2\pi - 2g'$.

Planetary Argument.	ν	ν'	l_s	l_c	l'_s	l'_c	$2e_s$	$2e_c$	$2e'_s$	$2e'_c$	$2e\pi_s$	$2e\pi_c$	$2e\pi'_s$	$2e\pi'_c$
V−g′	+.346	+.358	+1	0	−4	0	0	− 3	0	+22	− 4	0	+23	0
2V−2g′	+.341	+.364	−1	0	+6	0	0	+ 8	0	−31	+ 8	0	−31	0
2M−2g′	+.361	+.343	+2	0	−1	0	0	−11	0	+ 2	−11	0	+ 2	0
J−g′	+.361	+.344	+7	0	−1	0	−1	−39	0	+ 7	−38	+1	+ 7	0
2J−2g′	+.370	+.336	−3	0	−1	0	0	+18	0	+ 5	+18	0	+ 5	0

LUNAR ARGUMENT $2\lambda - 2D$.

	γ_s	γ_c	γ'_s	γ'_c		γ_s	γ_c	γ'_s	γ'_c
V−g′	0	+13	0	− 5	4M−3g′	−1	− 1	−1	+ 1
2V−3g′	+1	+ 7	+1	− 4	4M−2g′	+1	0	+1	+14
2V−2g′	0	−14	0	+17	J−2g′	−1	0	0	− 1
3V−4g′	+1	+ 4	0	− 1	J−g′	0	+11	+1	−19
3V−5g′	+1	+ 2	+2	− 2	J	+6	+ 1	+6	0
4V−4g′	0	0	0	−24	2J−3g′	0	0	0	0
M−g′	0	+ 1	0	− 1	2J−2g′	0	+25	0	+ 7
2M−2g′	0	+ 3	0	− 5	2J−g′	−1	− 2	−2	+ 4
2M−g′	0	− 3	0	− 4	3J−3g′	0	− 6	0	+ 2
3M−3g′	0	0	0	0	3J−2g′	−3	0	+1	0
3M−2g′	−1	− 1	−1	+ 1	3J−g′	0	0	+1	0

§ 76. *Reduction to inequalities of true longitude and collection of results.* To complete the work it is necessary to transform the elemental inequalities into inequalities of the coördinates. As already remarked, the parallax appears to contain no sensible terms arising from the action of the planets; only inequalities of longitude and latitude are therefore considered. In the case of terms of very long period the transformation to true longitude is unnecessary, because these terms can best be used and compared as elemental inequalities. A precise classification can not, however, be made between the terms which are to be transformed and those which are not. What has actually been done is to retain as elemental inequalities those depending on the longitude of the Moon's node, because though they may ultimately be transformed for use into the inequalities of the coördinates, they are to be combined with terms arising from the compression of the Earth having the same argument. The two Venus-terms of very long period have not been transformed because, as already remarked, they can be most conveniently applied to the elements. To transform the other terms put δv, the perturbations in longitude in orbit. Then

$$v = l + 2e \sin g + \tfrac{5}{4}e^2 \sin 2g$$

$$\delta v = \delta l + 2\delta e \sin g + \tfrac{5}{2}e\delta e \sin 2g + 2e\delta g \cos g + \tfrac{5}{2}e^2\delta g \cos 2g$$

Substituting

$$\delta l = l_c \cos G + l_s \sin G \qquad \delta\pi = \pi_c \cos G + \pi_s \sin G \qquad \delta e = e_c \cos G + e_s \sin G$$

$$\delta l - \delta\pi = \delta g = g_c \cos G + G_s \sin G$$

we shall have

$$\begin{aligned}
\delta v = \delta l &+ 2e_s \sin G \sin g + 2e_c \cos G \sin g \\
&+ 2eg_c \cos G \cos g + 2eg_s \sin G \cos g \\
&+ \tfrac{5}{2}ee_s \sin G \sin 2g + \tfrac{5}{2}ee_c \cos G \sin 2g \\
&+ \tfrac{5}{2}e^2g_c \cos G \cos 2g + \tfrac{5}{2}e^2g_s \sin G \cos 2g \\
= \delta l &- (e_s - eg_c) \cos (G + g) + (e_c + eg_s) \sin (G + g) \\
&+ (e_s + eg_c) \cos (G - g) - (e_c - eg_s) \sin (G - g) \\
&- \tfrac{5}{4}e(e_s - eg_c) \cos (G + 2g) + \tfrac{5}{4}e(e_c + eg_s) \sin (G + 2g) \\
&+ \tfrac{5}{4}e(e_s + eg_c) \cos (G - 2g) - \tfrac{5}{4}e(e_c - eg_s) \sin (G - 2g)
\end{aligned}$$

In nearly or quite all cases we may drop terms of the second order in e and use

$$\begin{aligned}
\delta v = l_c \cos G + l_s \sin G &+ [e(l_c - \pi_c) - e_s] \cos (G + g) + [e(l_s - \pi_s) + e_c] \sin (G + g) \\
&+ [e(l_c - \pi_c) + e_s] \cos (G - g) + [e(l_s - \pi_s) - e_c] \sin (G - g)
\end{aligned}$$

The subsequent processes are so simple and familiar as to scarcely need statement. All terms of δv depending on the same argument are combined into two, one depending on the sine, the other on the cosine of the argument. Their values are shown for each argument in the following table. The two terms are then combined into a monomial satisfying the equation

$$v_s \sin G + v_c \cos G = \delta v \sin (G + A)$$

Terms of which the coefficient δv was less than 0''.003, have generally, but not always, been dropped. It will be seen that even exceeding this limit there are more than 150 periodic inequalities. These are so arranged that any one argument can, it is hoped, readily be found on a system which will be evident by a little examination.

The constituents of the arguments, including π, are all measured from the Earth's perihelion ($\pi = 99^\circ.5$). The secular variations of the coefficients of the periodic terms are omitted, because they can better be derived by varying the eccentricity of the Earth's orbit in the expressions for the inequalities due to the Sun's action.

PERIODIC INEQUALITIES OF THE TRUE LONGITUDE.

ACTION OF VENUS.

Argument.	v_c	v_s	δv	A
	″	″	″	°
$v-2g'$	−.001	+.055	.055	359.0
$v-g'$	+.001	−.882	.882	179.9
v	−.000	−.014	.014	184.0
$2v-4g'$	−.004	+.003	.005	306.8
$2v-3g'$	+.096	−.340	.354	164.3
$2v-2g'$	+.005	+.401	.401	0.7
$3v-5g'$	+.060	−.056	.082	133.0
$3v-4g'$	+.040	−.191	.197	168.3
$3v-3g'$	+.001	−.037	.037	178.4
$5v-8g'$	+.018	−.015	.023	129.8
$g-5v+8g'$	+.004	+.003	.005	53.2
$g-3v+3g'$	.000	+.008	.008	0.0
$g-3v+4g'$	+.008	+.040	.041	11.3
$g-3v+5g'$	+.015	+.018	.023	39.8
$g-2v+2g'$	+.001	−.093	.093	180.0
$g-2v+3g'$	+.018	+.080	.082	12.6
$g-v$	.000	−.004	.004	180.0
$g-v+g'$	.000	+.166	.166	0.0
$g-v+2g'$	.000	−.003	.003	180.0
$g+v-2g'$	.000	+.003	.003	0.0
$g+v-g'$	.000	−.149	.149	180.0
$g+v$	.000	−.004	.004	180.0
$g+2v-3g'$	+.018	−.078	.080	167.0
$g+2v-2g'$	+.001	+.071	.071	0.6
$g+3v-5g'$	+.015	−.016	.022	136.9
$g+3v-4g'$	+.008	−.036	.037	167.5
$g+3v-3g'$	.000	−.006	.006	180.0
$g+5v-8g'$	+.004	−.003	.005	126.8
$2\pi-2v$	.000	−.009	.009	180.0
$2\pi-2v+g'$	+.001	+.003	.003	18.4
$2\pi-v-g'$	.000	+.007	.007	0.0
$2\pi+2v-4g'$	.000	+.008	.008	0.0
$2\pi+3v-5g'$	.000	+.072	.072	0.0
$2\pi+6v-10g'$	−.037	+.050	.062	323.5

Argument.	v_c	v_s	δv	A
	″	″	″	°
$g+2\pi-3v+g'$	.000	−.014	.014	180.0
$g+2\pi-3v+2g'$	+.012	+.051	.053	13.2
$g+2\pi-3v+3g'$	+.014	+.016	.021	41.2
$g+2\pi-2v$	+.001	−.174	.174	180.0
$g+2\pi-2v+g'$	+.011	+.070	.071	9.0
$g+2\pi-v-g'$	.000	+.146	.146	0.0
$g+2\pi+v-3g'$	.000	−.025	.025	180.0
$g+2\pi+2v-5g'$	+.011	−.040	.041	164.5
$g+2\pi+2v-4g'$	.000	+.142	.142	0.0
$g+2\pi+3v-7g'$	+.017	−.019	.026	138.2
$g+2\pi+3v-5g'$	+.002	−.646	.646	179.8
$g+2\pi+5v-8g'$	+.005	−.024	.025	168.3
$g+2\pi+6v-10g'$	−.051	+.066	.083	322.3
$2g+2\pi-3v+g'$	.000	−.035	.035	180.0
$2g+2\pi-3v+2g'$	+.019	+.090	.092	11.9
$2g+2\pi-3v+3g'$	+.016	+.020	.026	38.7
$2g+2\pi-2v$	+.003	−.142	.143	178.8
$2g+2\pi-2v+g'$	+.020	+.100	.102	11.3
$2g+2\pi-v-g'$	.000	+.096	.096	0.0
$2g+2\pi+v-3g'$	.000	−.008	.008	180.0
$2g+2\pi+2v-5g'$	+.010	−.048	.049	168.3
$2g+2\pi+2v-4g'$	.000	+.024	.024	0.0
$2g+2\pi+3v-7g'$	+.022	−.027	.035	140.8
$2g+2\pi+3v-6g'$	+.001	−.005	.005	168.7
$2g+2\pi+3v-5g'$	−.001	+.056	.056	359.0
$3g+2\pi-2v$	.000	+.009	.009	0.0
$3g+2\pi-v-g'$	.000	−.006	.006	180.0
$4g+2\pi-2v$	.000	+.003	.003	0.0
$4g+2\pi-v-g'$	.000	−.002	.002	180.0
$2\pi-g+3v-5g'$	.000	+.003	.003	0.0
$2\pi-g+6v-10g'$	−.002	+.002	.003	315.0

Action of Mars.

Argument.	v_c	v_s	δv	A	Argument.	v_c	v_s	δv	A
	″	″	″	°		″	″	″	°
$M-g'$	+.001	−.011	.011	174.3	$2\pi-2M-g'$	.000	+.002	.002	0.0
M	−.001	−.030	.030	181.9					
$2M-2g'$	.000	−.224	.224	180.0	$2\pi-2M-4g'$	.000	−.004	.004	180.0
$2M-g'$	−.193	+.317	.372	328.7					
$3M-3g'$	.000	+.014	.014	0.0	$g+2\pi-4M$	+.011	−.001	.011	95.2
$3M-2g'$	−.031	+.029	.042	313.1	$g+2\pi-4M+g'$	−.004	−.003	.005	233.2
$4M-3g'$	−.035	+.032	.047	312.4	$g+2\pi-3M$	−.004	−.004	.006	225.0
$4M-2g'$	+.095	+.002	.095	88.8	$g+2\pi-3M+g'$	.000	+.006	.006	0.0
$5M-3g'$	−.020	+.023	.030	319.0	$g+2\pi-2M+g'$	−.026	−.034	.043	217.4
					$g+2\pi-2M$	.000	+.017	.017	0.0
$g-4M+2g'$	+.017	−.002	.017	96.8	$g+2\pi-M-g'$	.000	+.004	.004	0.0
$g-4M+3g'$	−.007	−.007	.010	225.0	$g+2\pi+M-3g'$	.000	−.005	.005	180.0
$g-3M+2g'$	−.006	−.006	.009	225.0	$g+2\pi+2M-4g'$	.000	−.066	.066	180.0
$g-3M+3g'$	.000	−.003	.003	180.0	$g+2\pi+2M-3g'$	−.023	+.034	.041	325.9
$g-2M+g'$	−.035	−.042	.055	219.4	$g+2\pi+3M-5g'$	.000	+.003	.003	0.0
$g-2M+2g'$	.000	+.043	.043	0.0	$g+2\pi+3M-4g'$	−.006	+.006	.009	315.0
$g-M$	.000	+.003	.003	0.0	$g+2\pi+4M-5g'$	−.009	+.009	.013	315.0
$g-M+g'$	.000	+.004	.004	0.0	$g+2\pi+4M-4g'$	+.009	+.001	.009	83.7
$g+M-g'$	.000	−.004	.004	180.0					
$g+M$	.000	−.003	.003	180.0	$2g+2\pi-4M+g'$	−.004	−.003	.005	233.2
$g+2M-2g'$	.000	−.048	.048	180.0	$2g+2\pi-2M-g'$	−.035	−.046	.058	217.3
$g+2M-g'$	−.036	+.042	.055	319.4	$2g+2\pi-2M$	.000	+.006	.006	0.0
$g+3M-2g'$	−.006	+.006	.009	315.0	$2g+2\pi+2M-4g'$	.000	−.050	.050	180.0
$g+3M-3g'$	.000	+.003	.003	0.0	$2g+2\pi+2M-3g'$	−.030	+.042	.052	324.5
$g+4M-2g'$	+.017	+.002	.017	83.2	$2g+2\pi+4M-5g'$	−.020	+.019	.028	313.5
$g+4M-3g'$	−.007	+.007	.010	315.0	$3g+2\pi+2M-4g'$	.000	+.004	.004	0.0

ACTION OF JUPITER.

Argument.	v_c	v_s	δv	A	Argument.	v_c	v_s	δv	A
	″	″	″	°		″	″	″	°
$J-2g'$	+.004	−.015	.016	165.1	$g+2\pi-3J$	−.445	−0.015	0.445	268.0
$J-g'$	+.015	−.741	.741	178.8	$g+2\pi-3J+g'$	.000	+0.018	0.018	0.0
J	+.070	+.169	.183	22.5	$g+2\pi-2J-g'$	−.010	−0.016	0.019	212.0
$2J-3g'$	.000	+.006	.006	0.0	$g+2\pi-2J$	−.005	−1.140	1.140	180.2
$2J-2g'$	−.001	+.242	.242	359.8					
$2J-g'$	−.059	+.183	.193	342.2	$g+2\pi-J-2g'$	+.062	+0.008	0.062	82.7
$3J-3g'$	+.001	+.009	.009	6.4	$g+2\pi-J-g'$	.000	+0.064	0.064	0.0
$3J-2g'$	+.042	−.002	.042	177.2	$g+2\pi-J$	+.010	+0.015	0.018	33.7
$3J-g'$	−.024	+.005	.024	281.8	$g+2\pi+J-4g'$	+.001	−0.018	0.018	176.8
					$g+2\pi+J-3g'$	+.004	−0.230	0.230	179.0
					$g+2\pi+J-2g'$	+.060	−0.003	0.060	92.9
$g-3J+g'$	+.002	−.002	.003	135.0	$g+2\pi+2J-4g'$	+.001	+0.098	0.098	0.7
$g-3J+2g'$	+.010	.000	.010	90.0	$g+2\pi+2J-3g'$	−.018	+0.045	0.048	338.2
$g-3J+3g'$	.000	−.004	.004	180.0	$g+2\pi+3J-5g'$	+.001	+0.030	0.030	1.9
$g-2J+g'$	+.005	−.032	.032	179.1	$g+2\pi+3J-4g'$	+.021	−0.001	0.021	92.8
$g-2J+2g'$	.000	−.045	.045	180.0	$g+2\pi+3J-3g'$	+.004	+0.001	0.004	75.9
$g-J+g'$	+.004	+.140	.140	1.6					
$g-J+2g'$	+.001	+.006	.006	9.6	$2g+2\pi-3J$	−.007	0.000	0.007	270.0
$g-J$	+.036	+.011	.038	73.0	$2g+2\pi-3J+g'$	.000	−0.018	0.018	180.0
$g+J-2g'$	+.001	−.006	.006	170.4	$2g+2\pi-2J-g'$	−.008	−0.013	0.015	211.6
$g+J-g'$	+.004	−.163	.163	178.6	$2g+2\pi-2J$	.000	+0.018	0.018	0.0
$g+J$	+.036	−.011	.038	107.0	$2g+2\pi-J-2g'$	+.040	+0.006	0.040	81.5
$g+2J-2g'$	.000	+.064	.064	0.0	$2g+2\pi-J-g'$	.000	+0.016	0.016	0.0
$g+2J-g'$	+.005	+.036	.036	7.9	$2g+2\pi+J-4g'$	+.005	−0.038	0.038	172.5
$g+3J-3g'$	.000	+.003	.003	0.0	$2g+2\pi+J-3g'$	+.006	−0.168	0.168	177.9
$g+3J-2g'$	+.010	.000	.010	90.0	$2g+2\pi+J-2g'$	+.036	0.000	0.036	90.0
$g+3J-g'$	+.002	+.002	.003	45.0	$2g+2\pi+2J-5g'$	+.002	+0.019	0.019	6.0
					$2g+2\pi+2J-4g'$	−.001	+0.092	0.092	359.3
					$2g+2\pi+2J-3g'$	−.035	+0.082	0.089	336.9
$2\pi-3J$	−.258	−.008	.258	268.2	$2g+2\pi+3J-5g'$	+.003	+0.074	0.074	2.3
$2\pi-2J$	.000	+.256	.256	0.0	$2g+2\pi+3J-4g'$	+.042	−0.003	0.042	94.0
$2\pi-J-2g'$	+.004	.000	.004	90.0	$2g+2\pi+3J-3g'$	+.010	+0.002	0.010	78.5
$2\pi+J-3g'$	.000	−.011	.011	180.0	$3g+2\pi+J-3g'$	.000	+0.011	0.011	0.0
$2\pi+J-2g'$	+.004	.000	.004	90.0	$3g+2\pi+2J-4g'$	.000	−0.006	0.006	180.0
$2\pi+2J-4g'$	.000	+.004	.004	0.0	$2\pi-g-2J$	.000	+0.010	0.010	0.0
$2\pi+2J-3g'$	−.001	+.003	.003	341.6	$2\pi-g-3J$	−.015	−0.001	0.015	183.8

ACTION OF SATURN.

Argument.	v_c	v_s	δv	A	Argument.	v_c	v_s	δv	A
	″	″	″	°		″	″	″	°
S	+.016	+.048	.051	18.4	$g+S$	+.001	+0.004	0.004	14.1
$S-g'$	.000	−.040	.040	180.0	$g+S-g'$	.000	−0.008	0.008	180.0
$2S-g'$	−.001	+.011	.011	354.8					
$2S-2g'$	.000	+.008	.008	0.0	$g-S$	+.001	−0.004	0.004	165.9
					$g-S+g'$	.000	+0.008	0.008	0.0

§ 77. *Inequalities of the elements which have not been reduced to inequalities of the longitude.*

Mean longitude.

$$l = l_0 + nt + 5''.80\,T^2 + 0''.0020\,T^3 + 14''.77 \sin (18\text{v} - 16g' - g + 228°54')$$
$$+ 0''.247 \sin (8\text{v} - 13g' + 86°.4) + 0''.030 \sin \theta - 0''.273 \cos \theta$$

Longitude of Perigee.

$$\pi = \pi_0 + \pi_1 t + 253''.22\,T - 38''.49\,T^2 - 0''.013\,T^3 + 0''.47 \sin (18\text{v} - 16g' - g + 228°.5)$$
$$- 0''.67 \sin (8\text{v} - 13g' + 86°.4) - 0''.10 \sin \theta + 0''.80 \cos \theta$$

Longitude of Node.

$$\theta = \theta_0 + \theta_1 t - 137''.85\,T + 7''.62\,T^2 + 0''.0026\,T^3 + 2''.55 \sin \theta - 17''.33 \cos \theta \ (1800)$$
$$+ 2''.31 \sin \theta - 17''.36 \cos \theta \ (1900)$$

Sin ½ Inclination.

$$\delta\gamma = - 0''.115 \cos \theta - 0''.769 \sin \theta \ (1800)$$
$$- 0''.104 \cos \theta - 0''.770 \sin \theta \ (1900)$$

Hence:

Inclination.

$$\delta I = - 0''.230 \cos \theta - 1''.539 \sin \theta \ (1800)$$
$$- 0''.208 \cos \theta - 1''.541 \sin \theta \ (1900)$$

It may be found advisable, in the construction of new lunar tables, to include also the term

$$\delta l = 0''.256 \sin (2\pi - 2J)$$

in the mean longitude. The effect of including this term in the preceding transformations is that the Jovian evection, and the coefficient of the term of argument $2\pi - 2J - g$, have each received the increment $+ 0''.014$. Hence, if the term were included in the mean longitude, the coefficient of the Jovian evection would be $- 1''.154$, and of the other term named $- 0''.004$.

§ 78. ***Remarks on the Possibility of Unknown Terms of Long Period.*** In his ***Researches on the Motion of the Moon***, published in 1878,* the author found that the representation of the Moon's mean longitude during the period from 1650 to 1875 showed a discrepancy between existing theory and observation which might be represented by a term having a period of two or three centuries, and a coefficient of about 15''. This coefficient may be somewhat reduced by the introduction of the improved values of the terms of short period now available, but it does not seem likely that the deviation can be brought below 10''. One hypothesis on which the discrepancy might be explained is that of minute fluctuations in the

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* *Washington Observations for 1875*, App. II, p. 268. See also *Monthly Notices, Royal Astronomical Society*, vol. LXIII, March, 1903, p. 316.

Earth's diurnal rotation, which might be produced by the motion of solids and fluids on its surface. Observations of transits of Mercury leave scarcely more than a possibility of changes in the measure of time having the magnitude required to explain the deviation. The observed phenomena, therefore, point very strongly to the inference that there must be some term of long period still undiscovered in the actual mean motion of the Moon. The preceding researches seem to remove the possibility that there can be any undiscovered term in the action of the planets. It is true that there are two possible classes of inequality which are not considered in the present work. One of these has the solar parallax as a factor, and may arise from two sources; one the development of the potential to terms of higher order than the principal ones; the other to the parallactic terms in the Moon's coördinates. The author had intended to carry the development of R and Ω_p one step further, so as to include these terms. But, on examining the periods of the inequalities that might thus arise, none were found that could lead to any important term.

Yet another class of terms comprises those of the second order arising from the action of the planets being modified by their mutual perturbations. An examination which I believe to be exhaustive was therefore made for terms of long period of this class. None have been found, and the writer believes that none can exist more important than one of $0''.018$ computed by Radau. This term has the argument $5S - 2J$ of the great inequality between Jupiter and Saturn. In this connection it may be again remarked that, in determining the action of Venus in the present work, the mutual perturbations of Venus and the Earth have been taken account of. But no change is thus produced except in the Hansenian term of long period.

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